

Exact impulsive gravitational waves in spacetimes of constant curvature

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Abstract

Exact solutions exist which describe impulsive gravitational waves propagating in Minkowski, de Sitter, or anti-de Sitter universes. These may be either nonexpanding or expanding. Both cases in each background are reviewed here from a unified point of view. All the main methods for their construction are described systematically: the Penrose “cut and paste” method, explicit construction of continuous coordinates, distributional limits of sandwich waves, embedding from higher dimensions, and boosts of sources or limits of infinite accelerations.

Attention is concentrated on the most interesting specific solutions. In particular, the nonexpanding impulsive waves that are generated by null multipole particles are described. These generalize the well-known Aichelburg–Sexl and Hotta–Tanaka monopole solutions. Also described are the expanding spherical impulses that are generated by snapping and colliding strings. Geodesics and some other properties of impulsive wave spacetimes are also summarized.

1 Outline and personal notes

In recent years, various aspects of exact solutions of Einstein’s equations which describe impulsive waves in spaces of constant curvature have been thoroughly investigated. However, these studies are mostly scattered in a great number of particular articles published in different journals. The first purpose of this essay is to collect and unify the principal results. We attempt to provide a brief but comprehensive and self-contained review of all impulsive gravitational waves which may propagate in Minkowski, de Sitter, or anti-de Sitter universe. We hope that such a summary may be useful for further studies in classical relativity and also in quantum gravity or string theory.

Nevertheless, the main motivation for our work was to pay a tribute to Professor J. Bičák on the occasion of his 60th birthday. Jiří Bičák’s lifelong devotion to the investigation of exact radiative spacetimes is famous. He has contributed to various specific topics, including those closely related to impulsive waves, which we survey here. But Professor Bičák not only wrote many contributions which will be mentioned below in the appropriate context. During the past decades he has also been an inexhaustible fountain-head of inspiration and a continuous source of encouragement — and not only to members of the Prague relativity group.

Even more personally, it was Jiří Bičák — my academic teacher — who in 1985 suggested the topic of my diploma thesis “Gravitational radiation in cosmology”. Later he carefully supervised my doctoral thesis “On exact radiative space-times with cosmological constant”. During all these years I have been learning fundamental issues concerning the theory of gravitational radiation from him. In particular, the geometrical and physical investigation of the Kundt and the Robinson–Trautman classes of exact gravitational waves later proved to be crucial for my understanding of possible impulses in spaces of constant curvature. Moreover, almost all of my recent works on impulsive solutions have emerged

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from a fruitful collaboration with Professor J. B. Griffiths from Loughborough University. And again, it was Jiří Bičák who established our mutual contacts and has been promoting and encouraging these during the last years ... It is thus only natural to dedicate this summarizing essay to him on the occasion of the exceptional anniversary of his birthday.

The present work is organized as follows. We shall first briefly describe the background spaces of constant curvature, present all (nontwisting) type N exact vacuum solutions with a cosmological constant, and “heuristically” introduce gravitational impulses as limits of corresponding sandwich waves. In the main section 3, we shall describe in detail the methods of construction of these impulsive waves — the Penrose “cut and paste” method, explicit construction of continuous coordinates, distributional limits of sandwich waves, geometrical embedding from higher dimensions, and boosts or limits of infinite acceleration of particular solutions. Using all these methods, both nonexpanding and expanding impulsive waves are constructed, as summarized at the end of section 3. The physically important particular solutions are presented in section 4, namely nonexpanding impulses generated by multipole particles, and expanding impulses generated by snapping or colliding strings. Finally, some other properties are also discussed, in particular the behaviour of geodesics.

2 Exposition

2.1 Spaces of constant curvature

The spaces of constant curvature — Minkowski, de Sitter, and anti-de Sitter spacetimes — which form backgrounds for the impulsive solutions investigated here, are the simplest exact solutions of Einstein’s vacuum field equations with vanishing, positive, and negative cosmological constant Λ , respectively. Yet, these are extremely important in contemporary theoretical physics. As Jiří Bičák writes in his recent substantial essay [1] in honour of Jürgen Ehlers: “These solutions have played a crucial role in many issues in general relativity and cosmology, and most recently, they have become important prerequisites on the stage of the theoretical physics of the ‘new age’, including string theory and string cosmology”. The de Sitter universe is the basic model of early quasi-exponential inflationary phase of expansion of the universe [2]. Moreover, according to the “cosmic no-hair” conjecture [3], it is locally the asymptotic state of many cosmological models with a positive Λ . Recently, the anti-de Sitter spacetime has also become a subject of intensive studies thanks to the Maldacena’s conjecture [4] which relates string theory in asymptotically anti-de Sitter space to a non-gravitational conformal field theory on the boundary at spatial infinity.

The spacetimes of constant curvature are highly symmetric: they admit ten Killing vector fields which correspond to the maximal number of isometries possible in a 4-dimensional spacetime. Also, they are conformally flat and can thus be understood as specific portions of the Einstein static universe. Their global conformal structures are however different for different signs of Λ (see below).

It is well-known that the de Sitter and anti-de Sitter spacetimes can be represented as the 4-dimensional hyperboloid

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + \epsilon Z_4^2 = \epsilon a^2, \quad a = \sqrt{3/|\Lambda|}, \quad (1)$$

in the flat 5-dimensional space with metric

$$ds_0^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + \epsilon dZ_4^2, \quad (2)$$

where $\epsilon = 1$ for $\Lambda > 0$ and $\epsilon = -1$ for $\Lambda < 0$. Obviously, the de Sitter spacetime has the topology $R^1 \times S^3$ (R^1 corresponding to the time), whereas the topology of the anti-de Sitter spacetime is $S^1 \times R^3$.

Various parameterizations of the hyperboloid (1) are known which introduce suitable coordinates for the de Sitter or the anti-de Sitter spacetime. For example, the expressions

$$\begin{aligned} Z_0 &= \frac{1}{\sqrt{2}}(\mathcal{U} + \mathcal{V}) / \left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right], \\ Z_1 &= \frac{1}{\sqrt{2}}(\mathcal{U} - \mathcal{V}) / \left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right], \end{aligned}$$

$$\begin{aligned}
Z_2 &= \frac{1}{\sqrt{2}}(\eta + \bar{\eta}) / \left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right] , \\
Z_3 &= \frac{-i}{\sqrt{2}}(\eta - \bar{\eta}) / \left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right] , \\
Z_4 &= a \left[1 - \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right] / \left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right] ,
\end{aligned} \tag{3}$$

i.e. inversely $\mathcal{U} = \sqrt{2}a(Z_0 + Z_1)/(Z_4 + a)$, $\mathcal{V} = \sqrt{2}a(Z_0 - Z_1)/(Z_4 + a)$, $\eta = \sqrt{2}a(Z_2 + iZ_3)/(Z_4 + a)$, give the line element for all spaces of constant curvature in the conformally flat unified form

$$ds_0^2 = \frac{2d\eta d\bar{\eta} - 2d\mathcal{U}d\mathcal{V}}{\left[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V}) \right]^2} . \tag{4}$$

This is de Sitter space when $\Lambda > 0$, anti-de Sitter space when $\Lambda < 0$, and Minkowski space when $\Lambda = 0$.

However, the most natural coordinates which cover the whole de Sitter hyperboloid (1) are

$$\begin{aligned}
Z_0 &= a \sinh(\tau/a) , \\
Z_1 &= a \cosh(\tau/a) \cos \chi , \\
Z_2 &= a \cosh(\tau/a) \sin \chi \sin \theta \cos \phi , \\
Z_3 &= a \cosh(\tau/a) \sin \chi \sin \theta \sin \phi , \\
Z_4 &= a \cosh(\tau/a) \sin \chi \cos \theta .
\end{aligned} \tag{5}$$

With these the line element (2) becomes

$$ds_0^2 = -d\tau^2 + a^2 \cosh^2(\tau/a) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] . \tag{6}$$

In this case, the privileged timelike observers have worldlines on which the space coordinates χ , θ and ϕ are constant. For this family, the 3-spaces $\tau = \text{const.}$ (which represent the universe at this instant) have the geometry of S^3 which contracts to a minimum size at $\tau = 0$ and then re-expand. The global conformal structure of the de Sitter spacetime becomes obvious if we introduce the conformal time η_E by $\sin \eta_E = 1/\cosh(\tau/a)$. The line element (6) then becomes $ds_0^2 = \Omega^2 ds_E^2$, where $\Omega = a/\sin \eta_E$, and $ds_E^2 = -d\eta_E^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)$ is the standard metric of the Einstein static universe. Therefore, the corresponding Penrose conformal diagram covers the coordinate ranges $\chi \in [0, \pi]$ and $\eta_E \in (0, \pi)$. The boundaries of η_E (i.e. $\tau = \pm\infty$) represent the conformal infinities \mathcal{J}^\pm . Obviously, these have a spacelike character.

The natural global coordinate system for the anti-de Sitter spacetime is obtained by

$$\begin{aligned}
Z_0 &= a \cosh(r) \sin(t/a) , \\
Z_4 &= a \cosh(r) \cos(t/a) , \\
Z_1 &= a \sinh(r) \cos \theta , \\
Z_2 &= a \sinh(r) \sin \theta \cos \phi , \\
Z_3 &= a \sinh(r) \sin \theta \sin \phi ,
\end{aligned} \tag{7}$$

with which the line element (2) becomes

$$ds_0^2 = -\cosh^2 r dt^2 + a^2 [dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2)] . \tag{8}$$

Again, the privileged timelike (non-geodesic) observers have worldlines r , θ and ϕ constants, their proper times are given by $\tau = t \cosh(r)$. However, the timelike slices $t = \text{const.}$ are 3-dimensional spaces of constant negative curvature. The conformal structure is seen by introducing $\eta_E = t/a$ and $\tan \chi = \sinh(r)$. Then $ds_0^2 = \Omega^2 ds_E^2$, where $\Omega = a/\cos \chi$, so that the conformal diagram of the anti-de Sitter spacetime is given by the range $\chi \in [0, \frac{\pi}{2})$, η_E arbitrary. Thus, contrary to the Minkowski or de Sitter universe, the conformal infinity \mathcal{J} corresponding to $\chi = \frac{\pi}{2}$ has a timelike character in the anti-de Sitter spacetime.

Fundamental geometrical properties of the spaces of constant curvature have been described in detail in [5] and in more recent works and reviews, see e.g. [6]. Note also that the Minkowski, de Sitter, and anti-de Sitter spacetimes are stable with respect to general, nonlinear (but “weak”) vacuum perturbations, as shown recently in [7], [8].

2.2 Exact type N Einstein spaces

Impulsive waves can (at least from a physical point of view) most naturally be understood as “distributional limits” of suitable families of waves with “sandwich” profiles, such that these have ever “shorter duration” but simultaneously become “stronger” in the limit. Impulsive gravitational waves can be intuitively understood in exactly this way. Incidentally, in our contributions [9], [10] with Jiří Bičák we investigated classes of exact radiative spacetimes, which may be used for the above construction. It will thus be useful and convenient for our later purposes to summarize the fundamental properties of these spacetimes here.

In particular, we studied all nontwisting Petrov-type N solutions of vacuum Einstein field equations with a cosmological constant. For type N spacetimes, the Debever–Penrose vector field k^α is quadruple and defines a privileged null congruence characterized by expansion $\Theta = \frac{1}{2}k^\alpha_{;\alpha}$, twist $\omega = \sqrt{\frac{1}{2}k_{[\alpha;\beta]}k^{\alpha;\beta}}$, and shear $|\sigma| = \sqrt{\frac{1}{2}k_{(\alpha;\beta)}k^{\alpha;\beta} - \Theta^2}$. The Bianchi identities and the Kundt–Thompson theorem for type N Einstein spaces give $\sigma = 0$. Considering the nontwisting case only, we are left with two classes:

- the *Kundt* class of *nonexpanding* gravitational waves ($\Theta = 0$), cf. [11], [12], Ch. 27 in [13],
- the *Robinson–Trautman* class of *expanding* gravitational waves ($\Theta \neq 0$), cf. [14], Ch. 24 in [13].

Hereafter, we denote the above Kundt class by $KN(\Lambda)$, and the Robinson–Trautman class by $RTN(\Lambda)$.

In [9] we described these spacetimes in detail from a geometrical point of view in a unified formalism, we presented invariant subclasses, and found relations between them. All solutions of the Kundt class $KN(\Lambda)$ can be written in suitable coordinates, first given in [12], as

$$ds^2 = 2 \frac{1}{p^2} d\xi d\bar{\xi} - 2 \frac{q^2}{p^2} dudv + F du^2, \quad (9)$$

where $p = 1 + \frac{1}{6}\Lambda\xi\bar{\xi}$, $q = (1 - \frac{1}{6}\Lambda\xi\bar{\xi})\alpha + \bar{\beta}\xi + \beta\bar{\xi}$, $F = \kappa(q^2/p^2)v^2 - (q^2/p^2)_{,u}v + (q/p)H$, $\kappa = \frac{1}{3}\Lambda\alpha^2 + 2\beta\bar{\beta}$, and $H = H(\xi, \bar{\xi}, u)$. The vacuum field equation is $p^2 H_{\xi\bar{\xi}} + \frac{1}{3}\Lambda H = 0$, which has an explicit general solution given by $H = (f_{,\xi} + \bar{f}_{,\bar{\xi}}) - \frac{1}{3}\Lambda(\bar{\xi}f + \xi\bar{f})/p$, where $f(\xi, u)$ is an *arbitrary* function (otherwise (9) is a pure radiation spacetime). As shown in [12], [9], there exist the following geometrically distinct canonical subclasses characterized by specific choices of the parameters α and β :

$$KN(\Lambda) \left\{ \begin{array}{l} \Lambda = 0 \left\{ \begin{array}{l} \kappa = 0 : PP \equiv KN(\Lambda = 0)[\alpha = 1, \beta = 0], \\ \kappa > 0 : KN \equiv KN(\Lambda = 0)[\alpha = 0, \beta = 1], \end{array} \right. \\ \\ \Lambda \neq 0 \left\{ \begin{array}{l} \kappa > 0 : KN(\Lambda)I \equiv KN(\Lambda)[\alpha = 0, \beta = 1], \\ \kappa < 0 : KN(\Lambda^-)II \equiv KN(\Lambda < 0)[\alpha = 1, \beta = 0], \\ \kappa = 0 : KN(\Lambda^-)III \equiv KN(\Lambda < 0)[\alpha = 1, \beta = \sqrt{-\frac{\Lambda}{6}}e^{i\omega(u)}] \\ \quad \rightarrow KN(\Lambda^-)III_K \equiv KN(\Lambda < 0)[\alpha = 1, \beta = \sqrt{-\frac{\Lambda}{6}}]. \end{array} \right. \end{array} \right.$$

For vanishing cosmological constant Λ there are two subclasses. The simpler one represents the well-known pp -waves denoted here as PP , for which $p = 1 = q$, $F = f_{,\xi} + \bar{f}_{,\bar{\xi}}$. This has been investigated by many authors (for details and references see section 21.5 in [13]). The second subclass with $\Lambda = 0$ is KN which is a special class of solutions discovered by Kundt [11] (see Ch. 27 in [13]). Interestingly, there is an asymmetry in the case $\Lambda \neq 0$: there exist *three* distinct subclasses of nonexpanding waves for $\Lambda < 0$, whereas there is only *one* such subclass for $\Lambda > 0$, namely $KN(\Lambda)I$. The reason is the condition $\kappa = \frac{1}{3}\Lambda\alpha^2 + 2\beta\bar{\beta}$ which for $\Lambda > 0$ excludes the cases $\kappa < 0$ and $\kappa = 0$. Note also that there exists a special subclass $KN(\Lambda^-)III_K$ of the $KN(\Lambda^-)III$ subclass such that the Debever–Penrose vector is the *Killing* vector. It was explicitly demonstrated [9] that this subclass is identical with an interesting family of solutions found by Siklos [15], and analyzed in detail in [16].

The class of expanding Robinson–Trautman solutions $RTN(\Lambda)$ has been known for a long time [14]. In 1981, a more convenient coordinate parametrization was found [17]

$$ds^2 = 2v^2 d\xi d\bar{\xi} + 2v\bar{A} d\xi du + 2vA d\bar{\xi} du + 2\psi dudv + 2(A\bar{A} + \psi B) du^2, \quad (10)$$

where $A = \epsilon \xi - v f$, $B = -\epsilon + \frac{1}{2}v(f_{,\xi} + \bar{f}_{,\bar{\xi}}) + \frac{1}{6}\Lambda v^2 \psi$, $\psi = 1 + \epsilon \xi \bar{\xi}$, and $\epsilon = -1, 0, +1$. Again, the solutions depend on an arbitrary function $f(\xi, u)$. The parameter ϵ is proportional to the Gaussian curvature of the 2-surfaces $u = \text{const}$. Thus, there are nine invariant subclasses of this class since the cases $\epsilon = +1, 0, -1$ and $\Lambda > 0, \Lambda = 0, \Lambda < 0$ are independent.

In the subsequent paper [10] we presented a physical interpretation of all the nontwisting type N Einstein spaces (9), (10) which we based on the study of the deviation of geodesics [18]. It was demonstrated that there always exists an (essentially unique) orthonormal frame tied to any timelike observer. For the $KN(\Lambda)$ spacetimes this is given by

$$\begin{aligned} e_{(0)}^\mu &= (\dot{v}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u}) , & e_{(1)}^\mu &= -\frac{p}{\sqrt{2}} \left(\frac{2\mathcal{R}e\dot{\xi}}{q^2\dot{u}}, 1, 1, 0 \right) , \\ e_{(2)}^\mu &= -\frac{p}{\sqrt{2}} \left(\frac{2\mathcal{I}m\dot{\xi}}{q^2\dot{u}}, i, -i, 0 \right) , & e_{(3)}^\mu &= -\left(\dot{v} - \frac{p^2}{\dot{u}q^2}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u} \right) , \end{aligned} \quad (11)$$

and for the $RTN(\Lambda)$ solutions by

$$\begin{aligned} e_{(0)}^\mu &= (\dot{v}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u}) , & e_{(1)}^\mu &= \frac{1}{\sqrt{2}v} \left(\frac{2v}{\psi\dot{u}} \mathcal{R}e\{v\dot{\xi} + A\dot{u}\}, -1, -1, 0 \right) , \\ e_{(2)}^\mu &= \frac{1}{\sqrt{2}v} \left(\frac{2v}{\psi\dot{u}} \mathcal{I}m\{v\dot{\xi} + A\dot{u}\}, -i, i, 0 \right) , & e_{(3)}^\mu &= -\left(\dot{v} + \frac{1}{\psi\dot{u}}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u} \right) . \end{aligned} \quad (12)$$

An invariant form of the equation of geodesic deviation with respect to the above interpretation frame, along any timelike geodesic in the $KN(\Lambda)$ and $RTN(\Lambda)$ spacetimes, is then given by

$$\begin{aligned} \ddot{Z}^{(1)} &= \frac{\Lambda}{3} Z^{(1)} - \mathcal{A}_+ Z^{(1)} + \mathcal{A}_\times Z^{(2)} , \\ \ddot{Z}^{(2)} &= \frac{\Lambda}{3} Z^{(2)} + \mathcal{A}_+ Z^{(2)} + \mathcal{A}_\times Z^{(1)} , \\ \ddot{Z}^{(3)} &= \frac{\Lambda}{3} Z^{(3)} . \end{aligned} \quad (13)$$

The amplitudes of the gravitational waves are given by $\mathcal{A}_+ = \mathcal{R}e\{\mathcal{A}\}$, $\mathcal{A}_\times = \mathcal{I}m\{\mathcal{A}\}$ such that

$$\mathcal{A} = -\frac{pq}{2}\dot{u}^2 f_{,\xi\xi\xi} \quad (14)$$

for the $KN(\Lambda)$ spacetimes, and

$$\mathcal{A} = -\frac{\psi}{2v}\dot{u}^2 f_{,\xi\xi\xi} \quad (15)$$

in the $RTN(\Lambda)$ spacetimes (evaluated along the geodesics). The equations (13) express relative accelerations of nearby free test particles in terms of their actual positions. The frame components of the displacement vector $Z^{(i)} \equiv e_\mu^{(i)} Z^\mu$ determine the invariant distance between the test particles. Similarly, $\ddot{Z}^{(i)} \equiv e_\mu^{(i)} (D^2 Z^\mu / d\tau^2)$ are relative accelerations. The system (13) enables us to draw the following physical conclusions:

- The particles move isotropically one with respect to the other if $\mathcal{A} = 0$, in which case no gravitational wave is present. Indeed, both the $KN(\Lambda)$ and $RTN(\Lambda)$ spacetimes are conformally flat vacuum for $f_{,\xi\xi\xi} = 0$, and therefore Minkowski ($\Lambda = 0$), de Sitter ($\Lambda > 0$), and anti-de Sitter ($\Lambda < 0$). These maximally symmetric, homogeneous, and isotropic spacetimes are thus natural backgrounds for other “non-trivial” $KN(\Lambda)$ and $RTN(\Lambda)$ radiative solutions.
- If $f_{,\xi\xi\xi} \neq 0$ then the amplitudes \mathcal{A} do not vanish. The particles are influenced by the wave, in a similar way as they are affected by a standard gravitational wave in Minkowski background. For $\Lambda \neq 0$, the influence of the wave is added to the (anti-)de Sitter isotropic expansion/contraction. Therefore, the $KN(\Lambda)$ and $RTN(\Lambda)$ metrics can be interpreted as exact gravitational waves which propagate in spaces of constant curvature.

- The wave has a transverse character as only motions in the plane $(\mathbf{e}_{(1)}, \mathbf{e}_{(2)})$ perpendicular to $\mathbf{e}_{(3)}$ are affected. The propagation direction is $\mathbf{e}_{(3)}$ and coincides with the projection of the Debever–Penrose vector on the hypersurface orthogonal to the observer’s four-velocity.
- There are two polarization modes of the wave, “+” and “×”, with the corresponding amplitudes \mathcal{A}_+ and \mathcal{A}_\times . Under rotations in the transverse plane they transform according to $\mathcal{A}'_+ = \cos 2\vartheta \mathcal{A}_+ - \sin 2\vartheta \mathcal{A}_\times$, $\mathcal{A}'_\times = \sin 2\vartheta \mathcal{A}_+ + \cos 2\vartheta \mathcal{A}_\times$, so that the helicity of the wave is 2, as with linearized waves on a Minkowski background.
- Curvature singularities (which might be considered as “sources” of the gravitational waves) occur where the amplitudes \mathcal{A} diverge. Note that the singularities of the $RTN(\Lambda)$ spacetimes can be characterized by the non-vanishing invariant constructed recently by Bičák and Pravda [19] from the second derivatives of the Riemann tensor.
- Special classes of explicit geodesics were also discussed in [10]. For a *positive* cosmological constant, the amplitudes \mathcal{A} decay exponentially fast, i.e. the *waves are damped*. The spacetimes locally approach the de Sitter universe, which is an explicit demonstration of the cosmic no-hair conjecture [3] under the presence of waves in exact model spacetimes. With Jiří Bičák we also discussed the validity of the cosmic no-hair conjecture in the Robinson–Trautman radiative spacetimes of Petrov type *II* [20].

2.3 Motivation: impulses as limits of exact sandwich waves

We have already indicated at the beginning of previous section that probably the most natural way how to introduce impulsive waves in spaces of constant curvature is to consider a distributional limit of sandwich waves of the above $KN(\Lambda)$ and $RTN(\Lambda)$ spacetimes. Indeed, all these solutions depend, in suitable coordinates (9), (10), on an arbitrary function $f(\xi, u)$ where the coordinate u is the “retarded time”. Thus, an arbitrary wave-profile can be prescribed within the above families of exact radiative solutions. In particular, we can consider sandwich waves for which the wave-profiles $d_\varepsilon(u)$ of the function $f(\xi, u) \equiv f(\xi) d_\varepsilon(u)$ are nonvanishing only on a *finite interval* of u (of the length ε around $u = 0$). In the context of the simplest class of pp -waves this has been introduced already in the late 50’s by Bondi, Pirani and Robinson [21].

Now, the distributional limit $d_\varepsilon(u) \rightarrow \delta(u)$, where δ is the Dirac delta distribution, results (at least formally) in a solution which clearly represents an impulsive gravitational wave localized on a single wave-front $u = 0$. Such a procedure, in which a sequence of sandwich pp -waves with profiles $d_\varepsilon(u)$ becomes infinitesimal in duration but unbounded in the amplitude as $\varepsilon \rightarrow 0$, was considered in [22], [23] and later elsewhere [24]. In this simplest case given by the metric (9) with $p = 1 = q$, $F = H(\xi, \bar{\xi}) d_\varepsilon(u)$, one directly obtains the metric

$$ds^2 = 2 d\xi d\bar{\xi} - 2 du dv + H \delta(u) du^2, \quad (16)$$

which is the well-know Brinkmann form of impulsive plane wave in Minkowski background [22], [25]. Analogously, more general nonexpanding impulsive waves of the Kundt class $KN(\Lambda)$ for $\Lambda \neq 0$, and expanding Robinson–Trautman impulsive waves of the $RTN(\Lambda)$ class can be constructed (see section 3.3 below for more details).

Mathematical problems and questions arise however with the above procedure. What is the meaning of the metric with distributional terms? What is the manifold structure? Usually, the metric coefficients are considered to be C^2 continuous functions. And even worse: in the limit of the $RTN(\Lambda)$ sandwich-wave spacetimes one obtains a metric which contains a term proportional to δ^2 . The product of the Dirac distributions is a mathematically ill-defined object within the Schwartz linear distribution theory. For this, it may be necessary to invoke the more advanced Colombeau nonlinear theory of generalized functions [26].

In view of these (and other) open questions, it is of primary importance to investigate the construction of impulsive-wave spacetimes by *various* methods and approaches. These may provide a deeper understanding of the impulsive solutions based on mathematically sound definitions, discover mutual relations, investigate specific properties, and elucidate their physical interpretation.

3 Main theme: methods of construction of impulsive waves

The history of studies of impulsive waves can be divided (roughly speaking) into three periods. During the first epoch, which culminated in classic works by Lichnerowicz and others [27], principal mathematical properties of possible shock or impulsive waves in general relativity were found and investigated. It was demonstrated that these must necessarily be localized on null hypersurfaces, across which the derivatives of the metric (the second or the first) are discontinuous. Nevertheless, explicit constructions of impulsive solutions and further investigation of their properties had not been performed until the fundamental contributions by Penrose [25] and Aichelburg and Sexl [28] appeared at the beginning of 1970's. During this “Golden Era” most of the different methods of construction occurred almost simultaneously. These methods, based on various approaches, are summarized in the following table:

<i>method of construction</i>	<i>basic characteristics</i>
“cut and paste”	general but only “formal”
continuous coordinates	general and explicit
limits of sandwich waves	general, explicit (problems for expanding impulses)
embedding	special (nonexpanding $\Lambda \neq 0$ impulses only), explicit
boosts/infinite acceleration	special (but physically important) and explicit

The last era, which can be called the “Renaissance”, started at the beginning of 1990's with a fundamental paper by Hotta and Tanaka [29]. By boosting the Schwarzschild–(anti-)de Sitter black hole to the speed of light they constructed (after previous “no-go” attempts by several authors) a nonexpanding impulse in the de Sitter and anti-de Sitter universes. Simultaneously, an interesting expanding impulsive-wave solution generated by a “snapping” cosmic string in Minkowski space was discovered and discussed by Gleiser and Pullin [30], Bičák [31], Nutku and Penrose [32], and Hogan [33], [34].

As a result of these and subsequent systematic investigations an almost complete picture has now emerged. This comprises both nonexpanding and expanding impulses in constant-curvature spaces with an arbitrary value of the cosmological constant. We will describe the classic and also more recent results here. We shall present these in the context of a unifying description of the main methods by which complete families of impulsive gravitational waves can be constructed. We will also mention the corresponding references and historical remarks in some more detail.

3.1 The “cut and paste” method

Let us start with an elegant geometrical method for constructing general nonexpanding (plane) and expanding (spherical) impulsive gravitational waves in Minkowski background. This was presented by Penrose in now classic works [22], [25]. His “cut and paste” approach, which is in some respects similar to that of Israel [35], is based on cutting the spacetime manifold \mathcal{M} along a suitable null hypersurface and then re-attaching the two pieces with a specific “warp”. The first case leads to impulsive pp -waves, the second case gives expanding impulsive waves. Recently, this approach has also been extended to constant-curvature backgrounds with a nonzero cosmological constant Λ [36]–[40].

3.1.1 Nonexpanding impulsive waves

In this case, the Penrose method is based on the removal of the null hypersurface $\mathcal{U} = 0$ from spacetime in the metric form (4)

$$ds_0^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V}}{[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})]^2}, \quad (17)$$

and re-attaching the halves $\mathcal{M}_-(\mathcal{U} < 0)$ and $\mathcal{M}_+(\mathcal{U} > 0)$ by making the identification with a “warp” in the coordinate \mathcal{V} such that

$$\left[\eta, \bar{\eta}, \mathcal{V}, \mathcal{U} = 0_- \right]_{\mathcal{M}_-} \equiv \left[\eta, \bar{\eta}, \mathcal{V} - H(\eta, \bar{\eta}), \mathcal{U} = 0_+ \right]_{\mathcal{M}_+}, \quad (18)$$

where $H(\eta, \bar{\eta})$ is an arbitrary function of η and $\bar{\eta}$ alone. It was shown in [25] that the condition (18) automatically guarantees that the Einstein field equations are satisfied everywhere including on $\mathcal{U} = 0$. However, impulsive components are introduced into the curvature tensor proportional to $\delta(\mathcal{U})$. These represent gravitational (plus possibly null-matter) impulsive waves.

In [25] Penrose considered only the Minkowski background space (4) with $\Lambda = 0$, in which case the impulsive surface $\mathcal{U} = 0$ is a plane. Impulsive pp -waves are thus obtained. However, the above method can easily be generalized to $\Lambda \neq 0$ cases. It was shown in [37] that exactly the same junction conditions (18) applied to a general background spacetime (17) introduce impulsive waves also in the de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) universes. However, the geometries of these impulses are different as the null hypersurface $\mathcal{U} = 0$, along which the spacetime is cut and pasted, is described by the 2-metric $d\sigma^2 = 2(1 + \frac{1}{6}\Lambda\eta\bar{\eta})^{-2}d\eta d\bar{\eta}$. This is obviously a 2-dimensional space of constant Gaussian curvature $K = \frac{1}{3}\Lambda$. When $\Lambda = 0$ the impulsive wave surface is a *plane*. For $\Lambda > 0$ it is a *sphere*, while for $\Lambda < 0$ it is a *hyperboloid*. For cases $\Lambda \neq 0$, the geometry of these impulsive spherical and hyperboloidal waves was described in detail in [41] using various convenient coordinate representations. It was also demonstrated that the wave surfaces are indeed nonexpanding.

Let us remark that the Penrose “cut and paste” approach to construction of nonexpanding impulsive waves is closely related to the Dray and ’t Hooft method of “shift function”, which is prescribed on the horizon of a suitable background space [42].

3.1.2 Expanding impulsive waves

In [25] Penrose demonstrated that his “cut and paste” method can be also used to construct expanding spherical impulsive (purely) gravitational waves. However, the junction conditions are somewhat more complicated than in the nonexpanding case. Instead of starting with the familiar form of Minkowski or (anti-)de Sitter spacetime (4), it is necessary to first perform the transformation

$$\mathcal{U} = \frac{Z\bar{Z}}{p}V - U, \quad \mathcal{V} = \frac{V}{p} - \epsilon U, \quad \eta = \frac{Z}{p}V, \quad (19)$$

where

$$p = 1 + \epsilon Z\bar{Z}, \quad \epsilon = -1, 0, +1. \quad (20)$$

With this, the constant curvature spacetimes take the following form

$$ds_0^2 = \frac{2(V/p)^2 dZ d\bar{Z} + 2dU dV - 2\epsilon dU^2}{[1 + \frac{1}{6}\Lambda U(V - \epsilon U)]^2}. \quad (21)$$

In these coordinates, the hypersurface $U = 0$ is a *null cone* (a sphere expanding with the speed of light) as $\eta\bar{\eta} - \mathcal{U}\mathcal{V} \equiv U(V - \epsilon U) = 0$. The spacetime can now be divided into two halves $\mathcal{M}_-(U < 0)$ inside the null cone, and $\mathcal{M}_+(U > 0)$ outside this. Now, the Penrose junction conditions prescribe the identification

$$\left[Z, \bar{Z}, V, U = 0_- \right]_{\mathcal{M}_-} \equiv \left[h(Z), \bar{h}(\bar{Z}), \frac{(1 + \epsilon h\bar{h})V}{(1 + \epsilon Z\bar{Z})|h'|}, U = 0_+ \right]_{\mathcal{M}_+} \quad (22)$$

of the points from the two re-attached parts across the impulsive sphere $U = 0$. In (22) an arbitrary function $h(Z)$ introduces a specific “warp” corresponding to one of all possible impulsive *vacuum* spacetimes of this type.

In [25] Penrose described the above “cut and paste” construction for the case of expanding spherical gravitational waves in a Minkowski background. He assumed the simplest case $\Lambda = 0$, $\epsilon = 0$. Generalization to impulsive spherical waves in the de Sitter and anti-de Sitter universe (with $\epsilon = 0$) was later found by Hogan [36]. In [34] he also introduced impulses in Minkowski spacetime with $\epsilon = +1$. The completely general form (21), (22) of the Penrose junction conditions has been recently presented in our contributions [38], [40].

Note finally that a similar (yet somewhat different) “cut and paste” approach was considered by Gleiser and Pullin [30] for constructing a specific solution which represents a spherical impulsive gravitational wave generated by a “snapping” cosmic string in Minkowski space. This particular solution will be discussed below in subsection 3.5.2.

3.2 Continuous coordinates

The Penrose “cut and paste” approach described in the previous section 3.1 is a general method. By prescribing the junction conditions (18) or (22) in (17) or (21), respectively, all nonexpanding and expanding impulsive gravitational waves in Minkowski, de Sitter, and anti-de Sitter universe can be constructed. Nevertheless, these formal identifications of points on both sides of the impulsive hypersurface do not immediately provide explicit metric forms of the complete spacetimes. It is thus of primary interest to find a suitable coordinate system for the above solutions in which the metric is explicitly continuous everywhere, including on the impulse. In the following we describe the construction of these privileged continuous coordinates.

3.2.1 Nonexpanding impulsive waves

We start again with the metric (4) which represents spaces of constant curvature. Let us perform the transformation $\mathcal{U} = U$, $\mathcal{V} = V + H + UH_{,Z}H_{,\bar{Z}}$, and $\eta = Z + UH_{,\bar{Z}}$, where $H(Z, \bar{Z})$ is an arbitrary real function. This yields the metric

$$ds_0^2 = \frac{2|dZ + U(H_{,Z}\bar{Z}dZ + H_{,\bar{Z}}Zd\bar{Z})|^2 - 2dUdV}{[1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV - UG)]^2}, \quad (23)$$

where $G = H - ZH_{,Z} - \bar{Z}H_{,\bar{Z}}$. We now consider (23) for $U > 0$ and combine this with the line element (4) in which we take $\mathcal{U} = U$, $\mathcal{V} = V$, $\eta = Z$ for $U < 0$. The resulting metric can be written as

$$ds^2 = \frac{2|dZ + U\Theta(U)(H_{,Z}\bar{Z}dZ + H_{,\bar{Z}}Zd\bar{Z})|^2 - 2dUdV}{[1 + \frac{1}{6}\Lambda(Z\bar{Z} - UV - U\Theta(U)G)]^2}, \quad (24)$$

where $\Theta(U)$ is the Heaviside step function. This is continuous across the null hypersurface $U = 0$ but the discontinuity in the derivatives of the metric introduces impulsive components in the Weyl and curvature tensors proportional to the Dirac distribution [37], $\Psi_4 = (1 + \frac{1}{6}\Lambda Z\bar{Z})^2 H_{,ZZ} \delta(U)$, $\Phi_{22} = [(1 + \frac{1}{6}\Lambda Z\bar{Z})^3 ((1 + \frac{1}{6}\Lambda Z\bar{Z})^{-1} H)_{,Z\bar{Z}} + \frac{1}{3}\Lambda H] \delta(U)$, in a natural tetrad. The metric (24) thus explicitly describes impulsive waves in de Sitter, anti-de Sitter or Minkowski backgrounds. For $\Lambda = 0$, the line element (24) reduces to the well-known Rosen form of impulsive pp -waves [25], [43], [44]. Note also that the continuous coordinate system for the particular Aichelburg–Sexl solution [28] (see (52) further in the text) was found by D’Eath [45] and used for analytic investigation of ultrarelativistic black-hole encounters. This was used for studies of high-energy scattering in quantum gravity on the Planck scale [46].

The complete transformation which relates (4) and (24) can be written as

$$\begin{aligned} \mathcal{U} &= U, \\ \mathcal{V} &= V + H\Theta(U) + U\Theta(U)H_{,Z}H_{,\bar{Z}}, \\ \eta &= Z + U\Theta(U)H_{,\bar{Z}}, \end{aligned} \quad (25)$$

which is discontinuous in the coordinate \mathcal{V} on $\mathcal{U} = 0$. From (25), using the fact that the coordinates U, V, Z are continuous, we obtain exactly the Penrose junction condition (18) for reattaching the two halves of the spacetime \mathcal{M}_- and \mathcal{M}_+ with a warp. Thus, the above procedure is indeed an *explicit* Penrose’s “cut and paste” construction of all nonexpanding impulsive gravitational waves.

Of course, the discontinuity in the complete transformation (25), which (formally) relates the continuous to the distributional form of impulsive solutions, causes some mathematical problems. However, it has been recently shown in [44], using the previous results [47], that (25) is in fact a *rigorous* example of a generalized coordinate transformation in the sense of Colombeau’s generalized functions (at least in the case of impulsive pp -waves). Moreover, it is possible to interpret this change of coordinates as the distributional limit of a family of smooth transformations which is obtained by a general regularization procedure, i.e. by considering the impulse as a limiting case of sandwich waves with an *arbitrarily* regularized wave profile. These results put the formal (“physical”) equivalence of both continuous and distributional forms of impulsive spacetimes on a solid ground.

3.2.2 Expanding impulsive waves

In this case, we perform a more involved transformation of the metric (4) given by

$$\begin{aligned}\mathcal{V} &= AV - DU, \\ \mathcal{U} &= BV - EU, \\ \eta &= CV - FU,\end{aligned}\tag{26}$$

where

$$\begin{aligned}A &= \frac{1}{p|h'|}, \quad B = \frac{|h|^2}{p|h'|}, \quad C = \frac{h}{p|h'|}, \\ D &= \frac{1}{|h'|} \left\{ \frac{p}{4} \left| \frac{h''}{h'} \right|^2 + \epsilon \left[1 + \frac{Z}{2} \frac{h''}{h'} + \frac{\bar{Z}}{2} \frac{\bar{h}''}{\bar{h}'} \right] \right\}, \\ E &= \frac{|h|^2}{|h'|} \left\{ \frac{p}{4} \left| \frac{h''}{h'} - 2 \frac{h'}{h} \right|^2 + \epsilon \left[1 + \frac{Z}{2} \left(\frac{h''}{h'} - 2 \frac{h'}{h} \right) + \frac{\bar{Z}}{2} \left(\frac{\bar{h}''}{\bar{h}'} - 2 \frac{\bar{h}'}{\bar{h}} \right) \right] \right\}, \\ F &= \frac{h}{|h'|} \left\{ \frac{p}{4} \left(\frac{h''}{h'} - 2 \frac{h'}{h} \right) \frac{\bar{h}''}{\bar{h}'} + \epsilon \left[1 + \frac{Z}{2} \left(\frac{h''}{h'} - 2 \frac{h'}{h} \right) + \frac{\bar{Z}}{2} \frac{\bar{h}''}{\bar{h}'} \right] \right\}.\end{aligned}\tag{27}$$

Here $h = h(Z)$ is an arbitrary function and the derivative with respect to its argument Z is denoted by a prime. With this, the metric (4) of a constant curvature space becomes

$$ds_0^2 = \frac{2|(V/p)dZ + U p \bar{H} d\bar{Z}|^2 + 2dU dV - 2\epsilon dU^2}{[1 + \frac{1}{6}\Lambda U(V - \epsilon U)]^2},\tag{28}$$

where

$$H(Z) = \frac{1}{2} \left[\frac{h'''}{h'} - \frac{3}{2} \left(\frac{h''}{h'} \right)^2 \right].\tag{29}$$

Similarly as in the nonexpanding case, we may now combine the line element (28) for $U > 0$ and attach this to the metric (21) for $U < 0$ (which was obtained from (4) by the transformation (19)). The resulting complete line element can thus be written in the form

$$ds^2 = \frac{2|(V/p)dZ + U \Theta(U) p \bar{H} d\bar{Z}|^2 + 2dU dV - 2\epsilon dU^2}{[1 + \frac{1}{6}\Lambda U(V - \epsilon U)]^2}.\tag{30}$$

This metric which was presented for a Minkowski background in [32]–[34], with a cosmological constant in [36], and in the most general form in [38], [40] is explicitly continuous everywhere, including across the null hypersurface $U = 0$. Again, the discontinuity in the derivatives of the metric yields impulsive components in the Weyl and curvature tensors, $\Psi_4 = (p^2 H/V) \delta(U)$, $\Phi_{22} = (p^4 H \bar{H}/V^2) U \delta(U)$, using a naturally adapted tetrad. The spacetime is clearly conformally flat everywhere except on the impulsive wave surface $U = 0$. It is a vacuum solution everywhere except on the wave surface at $V = 0$, and at possible singularities of the function $p^2 H$.

By comparing the transformations (19) at $U = 0_-$ with (26), (27) at $U = 0_+$, we obtain exactly the Penrose junction conditions (22). The above procedure is thus an *explicit* “cut and paste” construction of expanding spherical pure gravitational waves in spaces of constant curvature.

Note also that new continuous coordinates which generalize (30) for the case $\Lambda = 0$ were found recently [48] as an extension of previous results for spherical shock waves [49]. These contain an additional parameter which is related to acceleration of the coordinate system.

3.3 Limits of sandwich waves

We have already mentioned in the introductory section 2 that impulsive waves can be understood as distributional limits of appropriate sequence of sandwich waves in a suitable family of exact radiative spacetimes. In fact, this seems to be the most intuitive way of construction of impulsive waves, although mathematical difficulties occur with this approach in the case of expanding impulses of the Robinson–Trautman type.

3.3.1 Nonexpanding impulsive waves

It has been demonstrated in [50] that *all* nonexpanding impulses in Minkowski, de Sitter or anti-de Sitter universes can simply be constructed from the Kundt class $KN(\Lambda)$ of exact type N solutions (9) by considering the distributional form $H(\xi, \bar{\xi}) \delta(u)$ of the structural function H . These metrics can thus be written as

$$ds^2 = 2 \frac{1}{p^2} d\xi d\bar{\xi} - 2 \frac{q^2}{p^2} dudv + \left[\kappa \frac{q^2}{p^2} v^2 - \frac{(q^2)_{,u}}{p^2} v + \frac{q}{p} H(\xi, \bar{\xi}) \delta(u) \right] du^2, \quad (31)$$

where $p = 1 + \frac{1}{6}\Lambda\xi\bar{\xi}$, $q = (1 - \frac{1}{6}\Lambda\xi\bar{\xi})\alpha + \bar{\beta}\xi + \beta\bar{\xi}$, and $\kappa = \frac{1}{3}\Lambda\alpha^2 + 2\beta\bar{\beta}$. As indicated in section 2.2, for a general wave-profile of the function H there exist various distinct canonical subclasses of (9) characterized by specific choices of the parameters α and β . However, *impulsive* limits of these subclasses become (locally) *equivalent*. Indeed, in the case of vanishing cosmological constant Λ there are two subclasses, namely the *pp*-waves PP and the Kundt subclass KN . However, the transformation

$$\mathcal{U} = (\xi + \bar{\xi})(1 + uv)u, \quad \mathcal{V} + 1 = (\xi + \bar{\xi})v, \quad \eta = \xi + (\xi + \bar{\xi})uv, \quad (32)$$

converts the impulsive KN metric (31) with $\alpha = 0$, $\beta = 1$ to the impulsive PP metric with $\alpha = 1$, $\beta = 0$. The only non-trivial impulsive gravitational waves of the form (31) in Minkowski space are thus impulsive *pp*-waves (16).

Similar results hold also for the $\Lambda \neq 0$ case. Generically, there are three distinct subclasses of nonexpanding waves $KN(\Lambda)I$, $KN(\Lambda^-)II$, and $KN(\Lambda^-)III$ (see section 2.2). It was shown in [50] that although these canonical subclasses are different for smooth profiles, they are equivalent for impulsive profiles. For example, the metric of the $KN(\Lambda^-)I$ subclass of (31) given by $\alpha = 0$, $\beta = 1$, $\Lambda < 0$ can be expressed, using the transformation $\xi = \sqrt{2}a e^{i\phi_1} \tanh(R_1/2)$, $v = \frac{1}{\sqrt{2}}a/t_1$, $u = \frac{1}{\sqrt{2}}(\rho_1 - t_1)/a$, as

$$ds^2 = \frac{a^2}{t_1^2} \sinh^2 R_1 \cos^2 \phi_1 (d\rho_1^2 - dt_1^2) + a^2 (dR_1^2 + \sinh^2 R_1 d\phi_1^2) + \sinh R_1 \cos \phi_1 H(R_1, \phi_1) \delta(t_1 - \rho_1) (dt_1 - d\rho_1)^2. \quad (33)$$

Similarly, the transformation $\xi = \sqrt{2}a e^{i\phi_2} \tanh(R_2/2)$, $v = -a^2/\rho_2$, $u = t_2 - \rho_2$ brings the impulsive subclass $KN(\Lambda^-)II$ of the metric (31) with $\alpha = 1$, $\beta = 0$, $\Lambda < 0$ to the form

$$ds^2 = \frac{a^2}{\rho_2^2} \cosh^2 R_2 (d\rho_2^2 - dt_2^2) + a^2 (dR_2^2 + \sinh^2 R_2 d\phi_2^2) + \cosh R_2 H(R_2, \phi_2) \delta(t_2 - \rho_2) (dt_2 - d\rho_2)^2. \quad (34)$$

However, the two metrics (33) and (34) are equivalent under the transformation

$$t_1^2 = t_2^2 - \rho_2^2 (1 - \tanh^2 R_2 \cos^2 \phi_2), \quad \rho_1 = \rho_2 \tanh R_2 \cos \phi_2, \\ \cosh R_1 = \frac{t_2}{\rho_2} \cosh R_2, \quad \sin \Phi_1 = - \frac{\sinh R_2 \sin \phi_2}{\sqrt{t_2^2 \rho_2^{-2} \cosh^2 R_2 - 1}}. \quad (35)$$

In this way, by considering the above distributional limit (31) of the $KN(\Lambda)$ class (9) we obtain an explicit form of solutions which represent nonexpanding impulses. The metric (31) contains a single Dirac delta in the du^2 term, and obviously generalizes the well-known Brinkmann form of impulsive *pp*-waves (16) to which it reduces when $p = 1 = q$.

Interestingly, there exists yet another simple coordinate form of the complete family of nonexpanding impulses of the Kundt class $KN(\Lambda)$ which also contains a term with the Dirac delta. This is obtained from the continuous form of the impulsive-wave metric (24) by applying the transformation

(25). If we take into account the terms which arise from the derivatives of $\Theta(\mathcal{U})$, this transformation relates (24) to

$$ds^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V} + 2H(\eta, \bar{\eta}) \delta(\mathcal{U}) d\mathcal{U}^2}{[1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})]^2}, \quad (36)$$

which explicitly includes the impulse located on the wavefront $\mathcal{U} = 0$. This is a gravitational wave or an impulse of null matter in any spacetime of a constant curvature (4), depending on the specific form of the function H (see [37], [51] for more details). Again, in Minkowski background this is just the Brinkmann form of a general impulsive pp -wave (16). In fact, the metric (36) is conformal to (16). (Recall in this context that Siklos [15] proved that Einstein spaces conformal to pp -waves only occur when $\Lambda < 0$. The impulsive case is clearly a counter-example to this result.)

The explicit transformation between (31) and (36) for the $KN(\Lambda)I$ subclass is

$$\begin{aligned} \mathcal{U} &= \frac{(\xi + \bar{\xi})(1 + uv)u}{1 - \frac{1}{6}\Lambda(\xi + \bar{\xi})(1 + uv)u}, \\ \mathcal{V} &= \frac{(\xi + \bar{\xi})v + \frac{1}{6}\Lambda\xi\bar{\xi}}{1 - \frac{1}{6}\Lambda(\xi + \bar{\xi})(1 + uv)u} - 1, \\ \eta &= \frac{\xi + (\xi + \bar{\xi})uv}{1 - \frac{1}{6}\Lambda(\xi + \bar{\xi})(1 + uv)u}, \end{aligned} \quad (37)$$

which reduces to (32) in the case $\Lambda = 0$. Similar transformations hold also for the subclasses $KN(\Lambda^-)II$ and $KN(\Lambda^-)III$. Therefore, the full family of impulsive limits (31) of nonexpanding sandwich waves of the Kundt class $KN(\Lambda)$ is indeed equivalent to the distributional form of the solutions (36), and consequently to the continuous metric (24) obtained by the explicit “cut and paste” method.

3.3.2 Expanding impulsive waves

As has been argued in [38], the family of solutions for expanding impulsive spherical gravitational waves can be considered to be an impulsive limit of the class $RTN(\Lambda)$ of vacuum Robinson–Trautman type N solutions with a cosmological constant. It is convenient to consider these solutions in the García–Plebański [17] coordinates (10). Introducing a new coordinate w by $w = \psi v$, and assuming the impulsive limit $f \equiv f(\xi)\delta(u)$, we can express the above family of expanding impulses as

$$\begin{aligned} ds^2 &= 2\frac{w^2}{\psi^2} \left| d\xi - f\delta(u) du \right|^2 + 2 du dw + \left(\frac{1}{3}\Lambda w^2 - 2\epsilon \right) du^2 \\ &\quad + w \left[(f_{,\xi} + \bar{f}_{,\bar{\xi}}) - \frac{2\epsilon}{\psi}(f\bar{\xi} + \bar{f}\xi) \right] \delta(u) du^2, \end{aligned} \quad (38)$$

where $f(\xi)$ is an arbitrary function. Similarly as in the nonexpanding case (31), the spherical impulse in the form (38) is explicitly located on the null surface $u = 0$. For $u \neq 0$ the spacetime again reduces to the background of constant curvature. However, a difficult mathematical problem occurs in this case: the metric (38) contains the *product* of the Dirac distributions in the first term. This is not a well-defined concept in the linear theory of distributions, so that the metric (38) is only formal. Nevertheless, it demonstrates that impulsive limits of Robinson–Trautman type N vacuum spacetimes are equivalent to expanding impulsive gravitational waves (30). Indeed, the transformation

$$w = \frac{\tilde{w}}{1 + \frac{1}{6}\Lambda\tilde{u}(\tilde{w} - \epsilon\tilde{u})}, \quad u = \int \frac{d\tilde{u}}{1 - \epsilon\frac{1}{6}\Lambda\tilde{u}^2}, \quad (39)$$

converts (38) to

$$\begin{aligned} ds^2 &= \frac{1}{[1 + \frac{1}{6}\Lambda\tilde{u}(\tilde{w} - \epsilon\tilde{u})]^2} \left[2\frac{\tilde{w}^2}{\psi^2} \left| d\xi - f\delta(\tilde{u}) d\tilde{u} \right|^2 + 2 d\tilde{u} d\tilde{w} - 2\epsilon d\tilde{u}^2 \right. \\ &\quad \left. + \tilde{w} \left[(f_{,\xi} + \bar{f}_{,\bar{\xi}}) - \frac{2\epsilon}{\psi}(f\bar{\xi} + \bar{f}\xi) \right] \delta(\tilde{u}) d\tilde{u}^2 \right]. \end{aligned} \quad (40)$$

Dropping the tildes and performing the following discontinuous transformation

$$\begin{aligned} u &= U + \Theta(U) [U_{inv}(U, V, Z, \bar{Z}) - U] , \\ w &= V + \Theta(U) [V_{inv}(U, V, Z, \bar{Z}) - V] , \\ \xi &= Z + \Theta(U) [Z_{inv}(U, V, Z, \bar{Z}) - Z] , \end{aligned} \quad (41)$$

where the functions U_{inv} , V_{inv} , Z_{inv} are obtained by the composition of the inverse transformation to (19) with (26), we put (40) into the metric form of all expanding impulsive spherical waves expressed in the continuous coordinate system (30) (see [38] for more details). This relation is obvious for all $U \neq 0$ if we use the identity $u(w - \epsilon u) = U(V - \epsilon U)$. Keeping the distributional terms arising from Θ and its derivative, we formally obtain also the impulsive terms proportional to δ , with the identification $f \equiv Z_{inv} - Z$ (evaluated on $U = 0$).

Of course, much work is required before the metric (38) and the discontinuous transformation (41) can be fully accepted in a mathematically rigorous way. A natural tool appears to be the Colombeau theory of nonlinear generalized functions [26], in the context of which products of distribution can (in principle) be handled.

3.4 Embedding from higher dimensions

The complete class of *nonexpanding* impulsive waves in spaces of constant curvature with a *non-vanishing* cosmological constant Λ can alternatively be introduced using a convenient 5-dimensional formalism as metrics

$$ds^2 = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + \epsilon dZ_4^2 + H(Z_2, Z_3, Z_4) \delta(Z_0 + Z_1) (dZ_0 + dZ_1)^2 , \quad (42)$$

restricted by the constraint (1). Physically, this is an embedding of impulsive *pp*-waves (42) which propagate in 5-dimensional flat space onto the (anti-)de Sitter hyperboloid. The wave is absent when $H = 0$, in which case (42) with the constraint (1) reduces to the standard form of the de Sitter or the anti-de Sitter spacetime (2). It has been demonstrated in [52] that for a nontrivial H the metric represents impulsive waves propagating in the (anti-)de Sitter universe. Each impulse is located on the null hypersurface $Z_0 + Z_1 = 0$. Considering (1), this is given by

$$Z_2^2 + Z_3^2 + \epsilon Z_4^2 = \epsilon a^2 , \quad (43)$$

which is a nonexpanding 2-sphere in the de Sitter universe or a hyperboloidal 2-surface in the anti-de Sitter universe. The wave surfaces can naturally be parametrized as

$$Z_2 = a\sqrt{\epsilon(1 - z^2)} \cos \phi , \quad Z_3 = a\sqrt{\epsilon(1 - z^2)} \sin \phi , \quad Z_4 = a z . \quad (44)$$

Various 4-dimensional parametrizations of the solutions (42) on (1) can be considered. For example,

$$Z_0 = \frac{\mathcal{U} + \mathcal{V}}{\sqrt{2}\Omega} , \quad Z_1 = \frac{\mathcal{U} - \mathcal{V}}{\sqrt{2}\Omega} , \quad Z_2 + iZ_3 = \frac{\sqrt{2}\eta}{\Omega} , \quad Z_4 = a \left(\frac{2}{\Omega} - 1 \right) , \quad (45)$$

where $\Omega = 1 + \frac{1}{6}\Lambda(\eta\bar{\eta} - \mathcal{U}\mathcal{V})$, brings the metric to the previous form (36) with $H = \sqrt{2}H/(1 + \frac{1}{6}\Lambda\eta\bar{\eta})$. Other natural coordinates which parametrize (42) have been discussed in [41].

We have remarked in previous sections that the above metrics may describe impulsive gravitational waves and/or impulses of null matter. Purely gravitational waves occur when the vacuum field equation

$$(\Delta + \frac{2}{3}\Lambda)H = 0 \quad (46)$$

is satisfied [51], [53], [37], in which $\Delta \equiv \frac{1}{3}\Lambda\{\partial_z[(1 - z^2)\partial_z] + (1 - z^2)^{-1}\partial_\phi\partial_\phi\}$ is the Laplacian on the impulsive surface.

3.5 Boosts and limits of infinite acceleration

Let us finally outline yet another fundamental method for the construction of particular (but physically important) impulses in spaces of constant curvature. It is based on boosting suitable, initially static sources to the speed of light which yields specific nonexpanding impulsive waves. Similarly, limits of infinite acceleration of specific sources give special expanding impulsive solutions.

3.5.1 Nonexpanding impulsive waves

It was first demonstrated in 1971 by Aichelburg and Sexl in a classic paper [28] that a specific impulsive gravitational pp -wave solution can be obtained by boosting the Schwarzschild black hole to the speed of light, while its mass is reduced to zero in an appropriate way. Let us start with the metric

$$ds^2 = -dt^2 + dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + \Psi(dt^2 + dr^2), \quad (47)$$

where $\Psi = 2M/r$, which is the Schwarzschild (static and spherically symmetric) solution linearized for small values of M . We introduce Cartesian coordinates $x = r \sin \vartheta \cos \varphi$, $y = r \sin \vartheta \sin \varphi$, $z = r \cos \vartheta$, and perform a boost in the x -direction,

$$t = \frac{\tilde{t} + v\tilde{x}}{\sqrt{1-v^2}}, \quad x = \frac{\tilde{x} + v\tilde{t}}{\sqrt{1-v^2}}. \quad (48)$$

In the limit as $v \rightarrow 1$, the line element (47) takes the form

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + dy^2 + dz^2 + 2(d\tilde{t} + d\tilde{x})^2 \lim_{v \rightarrow 1} \frac{\Psi}{1-v^2}, \quad (49)$$

where $\Psi = 2M(x^2 + y^2 + z^2)^{-1/2}$ and x is given by (48). To evaluate this limit, we employ the identity

$$\lim_{v \rightarrow 1} \frac{1}{\sqrt{1-v^2}} \Psi(x) = \delta(\tilde{t} + \tilde{x}) \int_{-\infty}^{+\infty} \Psi(x) dx, \quad (50)$$

(see [29], [41]) which is valid in the distributional sense. It is also necessary to scale the parameter M in Ψ to zero such that $8M = b_0 \sqrt{1-v^2}$, where b_0 is a new constant. Obviously, we obtain an impulsive pp -wave metric (16),

$$ds^2 = -d\tilde{t}^2 + d\tilde{x}^2 + dy^2 + dz^2 + H \delta(\tilde{t} + \tilde{x})(d\tilde{t} + d\tilde{x})^2, \quad (51)$$

where $H = \frac{1}{2}b_0 \int_{-\infty}^{+\infty} (\rho^2 + x^2)^{-1/2} dx$, and $\rho^2 = y^2 + z^2$. The divergence in the integral can be removed by the transformation $\tilde{t} - \tilde{x} \rightarrow \tilde{t} - \tilde{x} - \frac{1}{2}b_0 \lim_{v \rightarrow 1} \log \left(\tilde{x} + v\tilde{t} - \sqrt{(\tilde{x} + v\tilde{t})^2 + 1 - v^2} \right)$, which gives

$$H = \frac{1}{2}b_0 \int_{-\infty}^{+\infty} [(\rho^2 + x^2)^{-1/2} - (1 + x^2)^{-1/2}] dx = -b_0 \log \rho. \quad (52)$$

This is the famous Aichelburg–Sexl solution [28] which represents an axially-symmetric impulsive gravitational wave in Minkowski space generated by a single null monopole particle located on $\rho = 0$. Using a similar approach, numbers of other specific impulsive waves in flat space have been obtained by boosting various spacetimes of the Kerr–Newman [54] or the Weyl family [55].

The method can be generalized to obtain impulses in spacetimes with $\Lambda \neq 0$. This was first done in 1993 by Hotta and Tanaka [29] who boosted the Schwarzschild–de Sitter solution to obtain a nonexpanding spherical impulsive gravitational wave generated by a pair of null monopole particles in the de Sitter background. They also described an analogous solution in the anti-de Sitter universe. Their main “trick” was to consider the boost in the 5-dimensional representation of the (anti-)de Sitter spacetime (1), (2) where the boost can explicitly (and consistently) be performed. One starts with the line element

$$ds^2 = -\left(1 - \epsilon r^2/a^2\right) dt^2 + \left(1 - \epsilon r^2/a^2\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + \Psi \left[dt^2 + \left(1 - \epsilon r^2/a^2\right)^{-2} dr^2 \right], \quad (53)$$

where $\Psi = 2M/r$. This is a perturbation of the de Sitter ($\epsilon = +1$) or anti-de Sitter ($\epsilon = -1$) space corresponding to the Schwarzschild–(anti-)de Sitter solution [56], linearized for small M . As $\Lambda \rightarrow 0$, the line element (53) reduces to the Schwarzschild-like perturbations of the Minkowski spacetime (47). For $M = 0$ the metric (53) is the (anti-)de Sitter spacetime in standard “static” coordinates. These parametrize the hyperboloid (1) by $Z_1 = r \sin \vartheta \cos \varphi$, $Z_3 = r \sin \vartheta \sin \varphi$, $Z_2 = r \cos \vartheta$, and

$$\begin{aligned} \text{for } \epsilon = +1 : \quad Z_0 &= \sqrt{a^2 - r^2} \sinh(t/a) , \quad Z_4 = \pm \sqrt{a^2 - r^2} \cosh(t/a) , \\ \text{for } \epsilon = -1 : \quad Z_0 &= \sqrt{a^2 + r^2} \sin(t/a) , \quad Z_4 = \sqrt{a^2 + r^2} \cos(t/a) . \end{aligned} \quad (54)$$

Writing (53) as $ds^2 = ds_0^2 + ds_1^2$, where ds_0^2 has the form (2), we may now express the perturbation as

$$ds_1^2 = a^2 \Psi \left[\left(\frac{Z_4 dZ_0 - Z_0 dZ_4}{Z_4^2 - \epsilon Z_0^2} \right)^2 + \frac{a^2}{r^2} \left(\frac{Z_0 dZ_0 - \epsilon Z_4 dZ_4}{Z_4^2 - \epsilon Z_0^2} \right)^2 \right] , \quad (55)$$

where $r = \sqrt{Z_0^2 + \epsilon(a^2 - Z_4^2)}$. Performing a boost in the Z_1 -direction,

$$Z_0 = \frac{\tilde{Z}_0 + v \tilde{Z}_1}{\sqrt{1 - v^2}} , \quad Z_1 = \frac{\tilde{Z}_1 + v \tilde{Z}_0}{\sqrt{1 - v^2}} , \quad (56)$$

the (anti-)de Sitter background ds_0^2 is, of course, invariant. Using again the identity (50) and rescaling the coefficient as $8M = b_0 \sqrt{1 - v^2}$, in the limit $v \rightarrow 1$ the term ds_1^2 takes the form

$$ds_1^2 = H \delta(\tilde{Z}_0 + \tilde{Z}_1) (d\tilde{Z}_0 + d\tilde{Z}_1)^2 , \quad (57)$$

where

$$H(Z_4) = \frac{1}{4} b_0 a^2 \int_{-\infty}^{+\infty} \frac{\epsilon(a^2 - Z_4^2) Z_4^2 + (a^2 + Z_4^2) x^2}{(Z_4^2 - \epsilon x^2)^2 [x^2 + \epsilon(a^2 - Z_4^2)]^{3/2}} dx . \quad (58)$$

We immediately observe that the structure of the boosted metric is exactly that of (42) with the impulse located on (43). Considering the parametrization (44) of this wave surface and substituting θ for x so that $x = a \sqrt{\epsilon(1 - z^2)} \cot \theta$, we explicitly evaluate the integral (58) as

$$H = \frac{1}{4} b_0 \int_0^\pi \frac{z^2 + \cos^2 \theta}{(z^2 - \cos^2 \theta)^2} \sin^3 \theta d\theta = b_0 \left(\frac{z}{2} \log \left| \frac{1+z}{1-z} \right| - 1 \right) . \quad (59)$$

This is the axially-symmetric Hotta–Tanaka solution [29]. Further details on boosting monopole (and also multipole) particles to the speed of light in the (anti-)de Sitter universe, the geometry of the nonexpanding wave surfaces, and discussion of various useful coordinates can be found in [41], [55]. In particular, it was demonstrated that although the impulsive wave surface is nonexpanding, for $\Lambda > 0$ this coincides with the horizon of the closed de Sitter universe. The background space contracts to a minimum size and then reexpand in such a way that the nonexpanding impulsive wave in fact propagates with the speed of light from the “north pole” of the universe across the equator to its “south pole”.

3.5.2 Expanding impulsive waves

As shown above, nonexpanding impulsive waves can be obtained by boosting particular (initially static) sources to the speed of light. Interestingly, specific expanding impulsive spherical gravitational waves can also be obtained by considering exact solutions representing accelerating sources and taking these to the limit in which the acceleration becomes unbounded. It was first realized by Jiří Bičák [31] (see also [57], with recent generalizations in [58], [59]) that expanding impulses can be obtained from boost-rotation symmetric solutions which represent gravitational field of uniformly accelerating objects. In the limit of infinite acceleration, these well-known explicit exact radiative spacetimes form spherical impulsive waves attached to conical singularities.

As Bičák writes in his essay [1], “the boost-rotation symmetric solutions representing uniformly accelerated objects ... play a unique role among radiative spacetimes since they are asymptotically flat, in the sense that they admit global smooth sections of null infinity. And as the only known radiative solutions describing finite sources they can provide expressions for the Bondi mass, the news function, or the radiation patterns in explicit forms.” Special solutions of this type have been studied for almost forty years, first by Bondi [60], Bonnor and Swaminarayan [61], Israel and Khan [62], and then by many others — but Jiří Bičák in particular. In fact, the investigation of various aspects of radiative spacetimes and fields with boost-rotation symmetry is his “most typical” topic. The analysis of radiative properties of the Bonnor–Swaminarayan solutions became the basis of Bičák’s Ph. D. thesis and of his first paper published abroad [63]. He realized that these belong to a wide class of boost-rotation symmetric solutions (of which, for example, the well-known *C*-metric is also a member) and that the boost symmetry is the *only* admissible second symmetry of axially symmetric, asymptotically flat spacetimes which admit gravitational radiation. He and his collaborators have studied the unique properties of these spacetimes, including their global structure, in a number of fundamental works, e.g. [63]–[70], [31], [57]. He has also summarized these results in many inspiring reviews [71]–[76], [1]. Altogether, the contributions concerning boost-rotation symmetric solutions represent almost a quarter of all Bičák’s original publications.

The boost-rotation symmetric spacetimes can be described by the line element

$$ds^2 = e^\lambda d\rho^2 + \rho^2 e^{-\mu} d\phi^2 + (\zeta^2 - \tau^2)^{-1} \left[e^\lambda (\zeta d\zeta - \tau d\tau)^2 - e^\mu (\zeta d\tau - \tau d\zeta)^2 \right], \quad (60)$$

where specific metric functions μ and λ depend only on $\zeta^2 - \tau^2$ and ρ^2 . In particular, the Bonnor–Swaminarayan solution [61] generally contains five arbitrary constants m_1, m_2, A_1, A_2 , and B which determine the masses and uniform accelerations of two pairs of particles, and the singularity structure on the axis of symmetry $\rho = 0$ (see e.g. [61], [72], [58]). It is possible to choose the constant B such that some sections of the axis are regular. Of particular interest are special cases described by Bičák, Hoenselaers and Schmidt [64], [65] which represent only *two* (Curzon–Chazy) particles¹ of mass m which are accelerating in opposite directions with acceleration A . For these solutions the metric functions in (60) can be written in a simple form

$$\begin{aligned} \mu &= -\frac{2m}{AR} + 4mA + B, \\ \lambda &= -\frac{m^2}{A^2 R^4} \rho^2 (\zeta^2 - \tau^2) + \frac{2mA}{R} (\rho^2 + \zeta^2 - \tau^2) + B, \end{aligned} \quad (61)$$

where $R = \frac{1}{2} \sqrt{(\rho^2 + \zeta^2 - \tau^2 - A^{-2})^2 + 4A^{-2} \rho^2}$. For $B = 0$, the axis $\rho = 0$ is regular between the symmetrically located particles but these are connected to infinity by two semi-infinite strings which cause their acceleration. For $B = -4mA$ there is a finite expanding strut between the particles.

The limit of infinite acceleration $A \rightarrow \infty$ of these solutions was investigated by Bičák [31], [57]. It is necessary to scale the mass parameter m to zero in such a way that the “monopole moment” $M_0 = -4mA$ remains constant. In this limit we obtain

$$\begin{aligned} \mu &= B - M_0, \\ \lambda &= B - \text{sign}(\rho^2 + \zeta^2 - \tau^2) M_0. \end{aligned} \quad (62)$$

The resulting spacetime is locally flat everywhere except on the sphere $\rho^2 + \zeta^2 = \tau^2$. It therefore describes an expanding spherical impulsive gravitational wave generated by two particles which move apart at the speed of light in the Minkowski background and are connected to infinity by two semi-infinite strings ($B = 0$), or each other by an expanding strut ($B = M_0$). Performing the transformation (see, e.g. [68]) $\rho = \frac{1}{2}(v - u)$, $\tau = \pm \frac{1}{2}(v + u) \cosh \chi$, $\zeta = \frac{1}{2}(v + u) \sinh \chi$, we put the solution (60), (62) into the standard form of boost-rotational symmetric spacetimes

$$ds^2 = \frac{1}{4}(v - u)^2 e^{-\mu} d\phi^2 + \frac{1}{4}(v + u)^2 e^\mu d\chi^2 - e^\lambda du dv. \quad (63)$$

¹In [58] we investigated systematically all the spacetimes with four accelerating particles. Their limits $A_i \rightarrow \infty$ are unphysical, with the only exception of a particular solution which represents a finite string/strut between the two outer particles which snaps/breaks at its midpoint and the two broken ends separate at the speed of light.

The metric functions are $\mu = -M_0$, $\lambda = [\Theta(uv) - \Theta(-uv)] M_0$ for the two receding strings, or $\mu = 0$, $\lambda = 2\Theta(uv)M_0$ for the expanding strut. In both cases μ is a constant, but there is a discontinuity in the otherwise constant value of λ with the step $2M_0$ on the null cone $uv = 0$. It is possible to perform a further transformation to coordinates in which the metric is continuous everywhere. Indeed, in the region where the functions μ and λ are constant, the transformation $U = -u e^{\lambda+\mu/2}$, $V = \frac{1}{2} v e^{-\mu/2}$, $\psi = \chi e^\mu$, brings (63) into the form

$$ds^2 = (V + \frac{1}{2} P U)^2 d\phi^2 + (V - \frac{1}{2} P U)^2 d\psi^2 + 2 dU dV , \quad (64)$$

where $P = e^{-(\lambda+\mu)}$. The solution for *two receding strings* can thus be written in the form (64) with

$$P = \Theta(-U) + \beta^2 \Theta(U) , \quad (65)$$

where $\beta = \exp(M_0)$. The metric (64), (65) is exactly that constructed by Gleiser and Pullin [30] by their “cut and paste” method. It represents an impulsive spherical gravitational wave propagating in the Minkowski universe. Outside the wave ($U > 0$), there are two receding strings characterized by a deficit angle $(1 - \beta)2\pi$, which can be interpreted as remnants of one cosmic string which “snapped”. However, as pointed out by Bičák [31], [77], the complete solution rather describes two semi-infinite strings approaching at the speed of light and separating again at the instant at which they collide. The complementary solution for an *expanding strut* is also given by (64) with

$$P = \beta^{-2} \Theta(-U) + \Theta(U) . \quad (66)$$

In this case there is a string with the deficit angle $(1 - \beta^{-1})2\pi$ in the locally Minkowski space inside the impulsive wave ($U < 0$).

Recall that there is an alternative continuous form of the above solutions. Introducing $Z = \frac{1}{\sqrt{2}}(\psi + i\phi)$, we convert the metric (64), (65) into the form $ds^2 = 2 |V dZ + U H d\bar{Z}|^2 + 2 dU dV$, where $H = -\frac{1}{2}\beta^2$. Performing another transformation of the metric (63) in the region $u > 0$, $v > 0$ as

$$\begin{aligned} U &= -\frac{uv}{u+v} \exp \left[\frac{\lambda}{2} - \chi e^{(\mu-\lambda)/2} \right] , \\ V &= \frac{1}{2} (u+v) \exp \left[\frac{\lambda}{2} + \chi e^{(\mu-\lambda)/2} \right] , \\ Z &= \frac{1}{\sqrt{2}} \frac{v-u}{v+u} \exp \left[-\chi e^{(\mu-\lambda)/2} + i\phi e^{-(\mu+\lambda)/2} \right] , \end{aligned} \quad (67)$$

we obtain $ds^2 = 2 V^2 dZ d\bar{Z} + 2 dU dV$. These two metrics for $U > 0$ and $U < 0$ can be matched continuously across the null cone $U = 0$. In fact, this is exactly a particular case of the metric (30) for $\Lambda = 0$, $\epsilon = 0$, and constant $H = -\frac{1}{2}\beta^2$, which describes impulsive spherical wave [25], [32], [33], [38].

Recently we analyzed [58] the limit of infinite acceleration of much larger class of boost-rotationally symmetric spacetimes which generalize the Bonnor–Swaminarayan solution. These explicit solutions were found by Bičák, Hoenselaers and Schmidt [65] and represent two uniformly accelerating particles with an *arbitrary multipole structure* attached to conical singularities (as in the two cases above). These solutions can be written as [58]

$$\begin{aligned} \mu &= 2 \sum_{n=0}^{\infty} M_n \frac{P_n}{(x-y)^{n+1}} + B - M , \\ \lambda &= -2 \sum_{k,l=0}^{\infty} M_k M_l \frac{(k+1)(l+1)}{(k+l+2)} \frac{(P_k P_l - P_{k+1} P_{l+1})}{(x-y)^{k+l+2}} \\ &\quad - \left(\frac{x+y}{x-y} \right) \sum_{n=0}^{\infty} \frac{M_n}{2^n} \sum_{l=0}^n \left(\frac{2}{x-y} \right)^l P_l + B , \end{aligned} \quad (68)$$

where the constants M_n represent the multipole moments, the argument of the Legendre polynomials P_n is $\alpha = (1 - xy)/(x - y)$, B is a constant,

$$M \equiv \sum_{n=0}^{\infty} \frac{M_n}{2^n} , \quad (69)$$

and for x , y and α in (68) one has to substitute from

$$x - y = 4A^2R, \quad x + y = 2A^2(\rho^2 + \zeta^2 - \tau^2), \quad \alpha = \frac{1}{2}(\rho^2 - \zeta^2 + \tau^2 + A^{-2})/R. \quad (70)$$

The axis is regular *between* the two particles if $B = 0$. In this case there are strings connecting the particles to infinity. The alternative situation with a strut between the particles (and the axis regular *outside*) is given by $B = M$. The axis is regular *everywhere* (except at the particles) if the combination of multipole moments satisfies the condition $M = 0$ and we assume $B = 0$. In this case the particles are self-accelerating [65]. Considering only the case $n = 0$ in (68), we recover the previous solution (61) for the accelerating monopole particles, with $M = M_0 = -4mA$.

Now, we consider the null limit $A \rightarrow \infty$ of the general class of solutions (68) in which all the multipole moments M_n are kept constant. Interestingly, in this limit we again obtain (62), only the parameter M_0 is now replaced by a more general parameter M introduced in (69). For the particular values of the constant B described above this solution describes a snapping cosmic string, or an expanding strut. The ends of the strings/strut move in opposite directions with the speed of light, generating an impulsive spherical gravitational wave. In the limit, the multipole structure of initial particles *disappears* and the solution is characterised by the single constant M only. Thus, the limit $A \rightarrow \infty$ of any uniformly accelerating multipole particle is identical to that of a monopole particle $M_0 = M$, as obtained originally in [31], [57]. Of course, the metric can also be represented in the coordinates (64) or (30) with $H = -\frac{1}{2}\beta^2$, the only difference is that the parameter β has a more general form

$$\beta = \exp\left(\sum_{n=0}^{\infty} \frac{M_n}{2^n}\right). \quad (71)$$

In the subsequent paper [59] we also investigated all possible null limits $A \rightarrow \infty$ of another important explicit class of solutions with boost-rotation symmetry, namely the well-know C -metric

$$ds^2 = -A^{-2}(x+y)^{-2} \left(F^{-1}dy^2 + G^{-1}dx^2 + Gd\phi^2 - Fdt^2 \right), \quad (72)$$

where $F = -1 + y^2 - 2mAy^3$, $G = 1 - x^2 - 2mA x^3$. It was shown already in 1970 by Kinnersley and Walker [78] that this metric represents two uniformly accelerating *black holes*, each of mass m . The acceleration A is caused either by a strut between the black holes or by two semi-infinite strings connecting them to infinity. Radiative and asymptotic properties were investigated in [79]. Bonnor [80] found an explicit transformation of (72) into a form (60). When $mA < 1/(3\sqrt{3})$ there are four possible spacetimes according to different ranges of the coordinates x and y [81], and the corresponding explicit functions λ , μ have more complicated forms. In [59] we investigated the limits $A \rightarrow \infty$ of all these possibilities. It was demonstrated that (scaling again the mass parameter m to zero such that mA remains constant) this limit is *identical* to the metric (64) of a spherical impulsive gravitational wave generated either by a snapping string (65), or an expanding strut (66), with

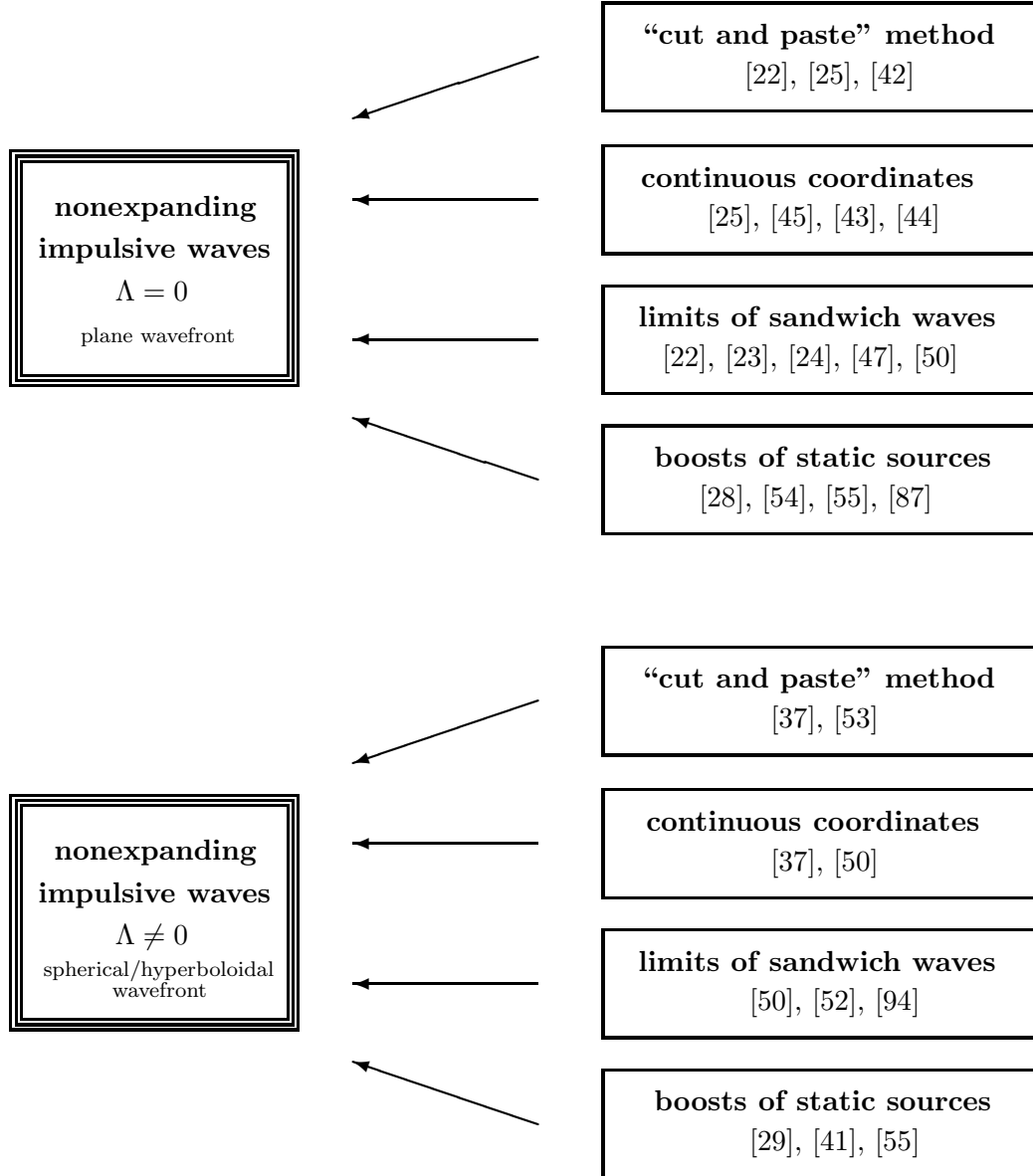
$$\beta = \frac{1}{2} [1 + \sqrt{3} \cot(\varphi + \frac{1}{3}\pi)] \in (0, 1], \quad \text{where } \varphi = \frac{1}{3} \arccos(1 - 54m^2A^2). \quad (73)$$

Note finally that it is natural to expect that the analogous null limit of infinite acceleration $A \rightarrow \infty$ of a more general C -metric, which admits a nonvanishing value of the cosmological constant Λ [82], would generate an expanding spherical impulsive wave (30) in the (anti-)de Sitter universe. However, such limit is mathematically more involved and has not yet been explicitly performed. We are currently investigating this problem. A first step has been achieved in [83] where a physical interpretation of the parameters of the solutions [82], which represent uniformly accelerating black holes in spacetimes of constant nonvanishing curvature, was presented (see also [84]). However, a deeper understanding of the global structure of these solutions is required. The most recent contribution by Bičák and Krtouš [85] may be of a great help.

3.6 Summary

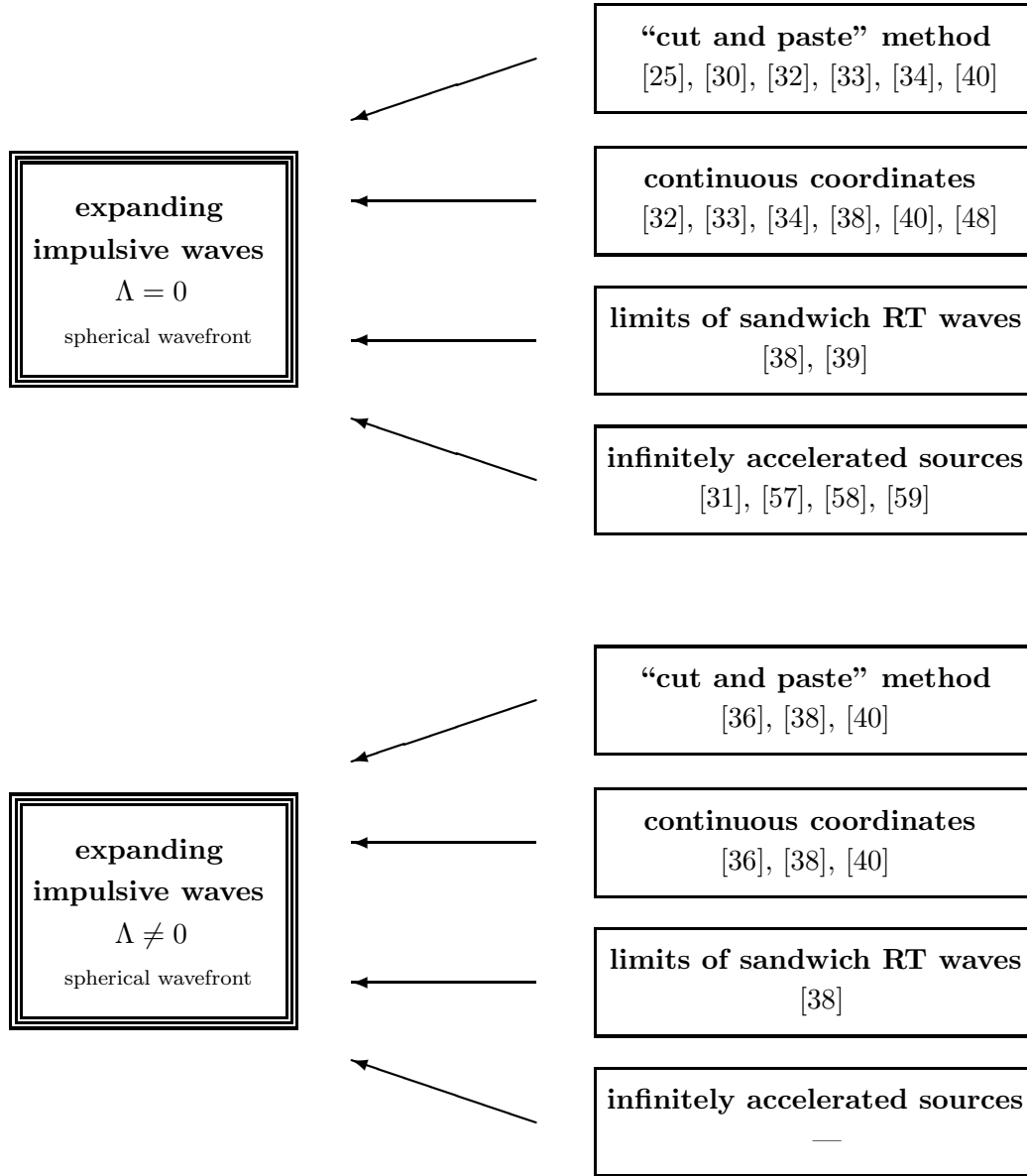
Let us now summarize for convenience all the methods described above for the construction of impulsive waves in spacetimes of constant curvature, together with the main corresponding references. This is presented in the following four diagrams, which separately describe the construction of nonexpanding/expanding waves in Minkowski/(anti-)de Sitter universes.

3.6.1 Nonexpanding impulsive waves



In addition, nonexpanding impulses with $\Lambda \neq 0$ can also be constructed by embedding impulsive pp -waves from higher dimensions on the (anti-)de Sitter hyperboloid, see the references [29], [41], [51], [52], [37].

3.6.2 Expanding impulsive waves



4 Particular solutions and some other properties

In the remaining part of this review some other properties of impulsive waves in spaces of constant curvature are briefly described. First, we present particular solutions of this type which are of interest from a physical point of view. Subsequently, we characterize geodesics and symmetries in these spacetimes.

4.1 Nonexpanding impulses generated by null multipole particles

It has been demonstrated above in section 3.5.1 that the simplest nonexpanding impulsive gravitational waves can be generated by boosting the Schwarzschild–(anti-)de Sitter black hole solutions. In the case $\Lambda = 0$ this gives the Aichelburg–Sexl solution (51), (52), and for $\Lambda \neq 0$ one obtains the Hotta–Tanaka solution (42), (59). Thus, the simplest impulsive solutions can be regarded as limits of static

“monopole” solutions boosted to the speed of light. This result can be generalized: it has been shown [55] that it is possible to consider particular nonexpanding impulsive waves generated by null *multipole* particles as limits of boosted static multipole particles.

It is well-known [13], [86] that static, axially symmetric and asymptotically flat vacuum solutions which represent external field of sources with a multipole structure can be written in the Weyl coordinates as

$$ds^2 = -e^{2\psi} dt^2 + e^{-2\psi} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2] , \quad (74)$$

where

$$\begin{aligned} \psi &= \sum_{m=0}^{\infty} \frac{a_m}{r^{m+1}} P_m(\cos \theta) , \\ \gamma &= \sum_{m,n=0}^{\infty} \frac{(m+1)(n+1)}{m+n+2} \frac{a_m a_n}{r^{m+n+2}} (P_{m+1} P_{n+1} - P_m P_n) , \end{aligned} \quad (75)$$

$r = \sqrt{\rho^2 + z^2}$, $\cos \theta = z/r$, and P_m are the Legendre polynomials with argument $\cos \theta$. Arbitrary constants a_m determine the m^{th} multipole moments of the source. Performing the boost (48) where $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, the line element (74) in the limit $v \rightarrow 1$ may be written in the form (49) with $\Psi = -2\psi$. Considering again the identity (50) and rescaling the parameters a_m to zero such that $8a_m = -m b_m \sqrt{1-v^2}$, with b_m being new constants which characterize the corresponding multipole moments of the boosted source, we obtain the impulsive pp -wave metric (51) where

$$H = \sum_{m=0}^{\infty} b_m H_m = \sum_{m=0}^{\infty} \frac{1}{2} m b_m \int_{-\infty}^{+\infty} \frac{1}{(\rho^2 + x^2)^{(m+1)/2}} P_m \left(\frac{z}{\sqrt{\rho^2 + x^2}} \right) dx . \quad (76)$$

For the simplest case $a_0 \neq 0$, $a_m = 0$ for $m \geq 1$ (which corresponds to the boosted Curzon–Chazy solution) the integral gives exactly the same result (52) as the Aichelburg–Sexl solution [28] generated by a single null monopole particle. For the higher multipole components $m \geq 1$ the integral (76) can also be explicitly evaluated. As shown in [55], we obtain

$$H_m(\rho, \phi) = \rho^{-m} \cos[m(\phi - \phi_m)] , \quad (77)$$

where $\rho^2 = y^2 + z^2$, $\cos(\phi - \phi_m) = z/\rho$, and ϕ_m is a constant. This term represents the m^{th} multipole component of the exact impulsive purely gravitational pp -wave generated by a source of an arbitrary multipole structure [87]. Indeed, the field equations for the metric (51) with

$$H = -b_0 \log \rho + \sum_{m=1}^{\infty} b_m \rho^{-m} \cos[m(\phi - \phi_m)] , \quad (78)$$

correspond to a source localized on the impulsive wavefront $u \equiv \tilde{t} + \tilde{x} = 0$ at $\rho = 0$, which is described by $T_{uu} = J(\rho, \phi) \delta(u)$ where

$$J(\rho, \phi) = \frac{1}{8} b_0 \delta(\rho) + \sum_{m=1}^{\infty} \frac{1}{8} b_m \frac{(-1)^m}{(m-1)!} \delta^{(m)}(\rho) \cos[m(\phi - \phi_m)] . \quad (79)$$

Clearly, the parameters b_m and ϕ_m represent the amplitude and phase of each multipole component.

Interestingly, there are analogous impulsive solutions also in the case $\Lambda \neq 0$. It was demonstrated in [52] that nontrivial solutions of the vacuum Einstein equation (46) for the metric (42) on (1), expressed in terms of the parameters (44), can be written as

$$H(z, \phi) = \sum_{m=0}^{\infty} b_m H_m = \sum_{m=0}^{\infty} b_m Q_1^m(z) \cos[m(\phi - \phi_m)] , \quad (80)$$

where $Q_1^m(z)$ are associated Legendre functions of the second kind generated by the relation

$$Q_1^m(z) = (-\epsilon)^m |1 - z^2|^{m/2} \frac{d^m Q_1(z)}{dz^m} . \quad (81)$$

The first term for $m = 0$,

$$Q_1(z) \equiv Q_1^0(z) = \frac{z}{2} \log \left| \frac{1+z}{1-z} \right| - 1, \quad (82)$$

exactly corresponds to the simplest axisymmetric Hotta–Tanaka solution (59). The higher components H_m describe nonexpanding impulsive gravitational waves in (anti-)de Sitter universe generated by null point sources with an m -pole structure, localized on the wave-front (43) at the singularities $z = \pm 1$, see [52]. The general solution (80) corresponds to a source described by

$$J(z, \phi) = \sum_{m=0}^{\infty} b_m \frac{\epsilon(-1)^m}{8\pi} (1-z^2)^{m/2} \left[\delta^{(m)}(z-1) + \delta^{(m)}(-z-1) \right] \cos[m(\phi - \phi_m)]. \quad (83)$$

Again, each component represents a point source with an m -pole structure, where the constants b_m give the strength, and ϕ_m the orientation of each m -pole.

To complete the picture, it would be natural to regard these explicit impulsive solutions (80) in the (anti-)de Sitter universe generated by *null* multipole particles as limits of boosted *static* multipole particles, as in the case of Minkowski background. Unfortunately, no explicit exact solutions are known which describe static sources of any multipole structure in a background with $\Lambda \neq 0$. Nevertheless, we were able to argue in [55] that it is possible to obtain the correct structure of the impulsive multipole solution (42), (80) by boosting the perturbation of the (anti-)de Sitter spacetime of the form (53) if the function $\Psi(t, r, \vartheta)$ has an appropriate form. The corresponding function H in (57) is given by

$$H(z, \phi) = a^3 [\epsilon(1-z^2)]^{3/2} \int_0^\pi \frac{z^2 + \cos^2 \theta}{(z^2 - \cos^2 \theta)^2} \frac{\Psi(t, r, \vartheta)}{r^2} d\theta, \quad (84)$$

in which the coordinates t, r, ϑ must be expressed using the relations

$$\cosh^2 \frac{t}{a} = \frac{z^2 \sin^2 \theta}{z^2 - \cos^2 \theta}, \quad r = \frac{a \sqrt{\epsilon(1-z^2)}}{\sin \theta}, \quad \cos \vartheta = \sin \theta \cos \phi, \quad (85)$$

(for $\Lambda < 0$ the function \cosh must be replaced by \cos). This is a generalization of the integral (59) which yields the Hotta–Tanaka solution in the monopole case $\Psi = 2M/r$.

4.2 Expanding impulses generated by snapping and colliding strings

A physically most interesting expanding spherical impulsive gravitational wave is probably that generated by a *snapping cosmic string*. This explicit solution, described in detail above in section 3.5.2., can be written in various forms — for example using the coordinates (64) with (65), or in the form (30) with a constant value $H = -\frac{1}{2}\beta^2$ (see the transformation (67) and the related text). Alternatively, the spacetime can also be written in the form (30) with the function H given by

$$H = \frac{\frac{1}{2}\delta(1 - \frac{1}{2}\delta)}{Z^2}, \quad (86)$$

where $\delta > 0$. This is generated from the function

$$h(Z) = Z^{1-\delta}, \quad (87)$$

by the expression (29). As shown in [32], [40], the generating function h is closely related to a geometrical interpretation of the Penrose junction conditions (22). If we evaluate the ratio η/\mathcal{V} using (19) for $U < 0$, and using (26) for $U > 0$, which is related to the explicit construction of a continuous coordinates (30), we observe that on the impulse $U = 0$

$$\frac{\eta}{\mathcal{V}} = \begin{cases} Z & \text{for } U = 0_- , \\ h(Z) & \text{for } U = 0_+ . \end{cases} \quad (88)$$

However, it follows from (4), (3) that $\eta/\mathcal{V} = (x + iy)/(t - z)$ in Minkowski space, or similarly $\eta/\mathcal{V} = (Z_2 + iZ_3)/(Z_0 - Z_1)$ in (anti-)de Sitter space. In both cases, this is exactly the relation for a

stereographic correspondence between a sphere and an Argand plane by projection from the North pole onto a plane through the equator [88]. This permits us to represent the wave surface $U = 0$ either as a Riemann sphere, or as its associated complex plane parametrized by the coordinate Z . The Penrose junction condition (22) across the wave surface then can be understood as a mapping on the complex Argand plane $Z \rightarrow h(Z)$. According to (88), this is equivalent to mapping points P_- on the “inside” of the wave surface to the identified points P_+ on the “outside”.

We may assume that the spacetime $U < 0$ *inside* the impulse represented by $Z = |Z|e^{i\phi}$ covers the *complete* sphere, $\phi \in [-\pi, \pi)$. However, the range of the function $h(Z)$ does not cover the entire sphere outside the spherical impulse for $U > 0$. In particular, the complex mapping (87) covers the plane minus a wedge as $\arg h(Z) \in [-(1-\delta)\pi, (1-\delta)\pi)$. This represents Minkowski, de Sitter, or anti-de Sitter space with a deficit angle $2\pi\delta$ which may be considered to describe a snapped cosmic *string* in the region *outside* the spherical wave. The strings have a constant tension and are located along the axis $\eta = 0$.

With the help of the above geometrical insight we were able to construct even more general explicit solutions, namely expanding impulsive waves generated by two *colliding cosmic strings* [40]. In such a situation, as first suggested by Nutku and Penrose [32], two non-aligned strings initially approach each other and snap at their common point at the instant at which they collide. The remnants are four semi-infinite strings which recede from the common point of interaction with the speed of light, generating a specific spherical impulse. As shown in [40], such a solution is described by the metric (30) with the structural function of the form

$$H = \frac{\frac{1}{2}\delta(1-\frac{1}{2}\delta)}{Z^2} - \frac{\frac{1}{2}\varepsilon(1-\frac{1}{2}\varepsilon)16w_1^2(1-\delta)^2Z^{-2\delta}}{[(w_1+i)Z^{1-\delta} + (w_1-i)]^2[(w_1-i)Z^{1-\delta} + (w_1+i)]^2}, \quad (89)$$

which is generated from

$$h(Z) = w_2 \frac{h_c^{1-\varepsilon} - 1}{h_c^{1-\varepsilon} + 1}, \quad \text{where} \quad h_c = -\frac{(w_1-i)Z^{1-\delta} + (w_1+i)}{(w_1+i)Z^{1-\delta} + (w_1-i)}. \quad (90)$$

Here the parameters δ and ε are the deficit angles of the first and the second pair of strings, respectively, and w_1, w_2 determine their velocities. For more details and illustrative pictures see [40].

Let us also mention that Hortaçsu and his collaborators [89] analyzed vacuum fluctuations and aspects of particle creation on the backgrounds of expanding impulsive waves.

4.3 Geodesics in impulsive spacetimes

In this final section we now mention some other properties of impulsive waves in spaces of constant curvature. In particular, we describe the behaviour of geodesics in these spacetimes.

4.3.1 Geodesics in nonexpanding impulsive waves

Geodesics in Minkowski space with plane-fronted impulsive *pp*-waves were discussed in many works, e.g. in [42], [53], [90]. However, as the corresponding geodesic and geodesic deviation equations in standard coordinates (16) contain highly singular products of distributions, an advanced framework of Colombeau algebras of generalized functions had to be employed to solve these equations in a mathematically rigorous sense [47].

In [91] we have studied the behaviour of geodesics in spacetimes which describe nonexpanding impulsive waves in the (anti-)de Sitter background. These recent results generalize those for impulsive *pp*-waves to the case of a nonvanishing cosmological constant Λ . The geodesic equations in such spacetimes can be derived conveniently using the embedding of the 5-dimensional *pp*-waves (42) onto the 4-dimensional hyperboloid (1). We obtain [91]

$$\begin{aligned} \ddot{U} &= -\frac{1}{3}\Lambda U e, \\ \ddot{V} - \frac{1}{2}H\delta'(U)\dot{U}^2 - H_{,p}\dot{Z}_p\delta(U)\dot{U} &= -\frac{1}{3}\Lambda V \left[e + \frac{1}{2}G\delta(U)\dot{U}^2 \right], \end{aligned} \quad (91)$$

$$\begin{aligned}\ddot{Z}_i - \frac{1}{2}H_{,i}\delta(U)\dot{U}^2 &= -\frac{1}{3}\Lambda Z_i \left[e + \frac{1}{2}G\delta(U)\dot{U}^2 \right] , \\ \ddot{Z}_4 - \frac{1}{2}\epsilon H_{,4}\delta(U)\dot{U}^2 &= -\frac{1}{3}\Lambda Z_4 \left[e + \frac{1}{2}G\delta(U)\dot{U}^2 \right] ,\end{aligned}$$

where $p = 2, 3, 4$, $i = 2, 3$, $e = 0, -1, +1$ for null, timelike or spacelike geodesics, respectively, and $G = Z_p H_{,p} - H$ (summation convention is used). The coordinates $U \equiv \frac{1}{\sqrt{2}}(Z_0 + Z_1)$ and $V \equiv \frac{1}{\sqrt{2}}(Z_0 - Z_1)$ are introduced here for convenience, and are different from those used in the metric (24). Also, we have rescaled the structural function as $\sqrt{2}H \rightarrow H$.

The equation for U in (91) is clearly decoupled and does not involve any distributional term. The solution, which is everywhere a *smooth* function of the affine parameter τ , can be written (without loss of generality) as

$$U = \tau , \quad U = a \dot{U}^0 \sinh(\tau/a) , \quad U = a \dot{U}^0 \sin(\tau/a) , \quad (92)$$

for $\epsilon e = 0$, $\epsilon e < 0$ or $\epsilon e > 0$, respectively. These relations allow us to take U as the geodesic parameter. Then a general solution of the remaining four functions Z_p and V in (91) can be written as

$$\begin{aligned}Z_p(U) &= Z_p^0 \sqrt{1 - \frac{1}{3}\Lambda e (\dot{U}^0)^{-2} U^2} + (\dot{Z}_p^0 / \dot{U}^0) U + A_p \Theta(U) U , \\ V(U) &= V^0 \sqrt{1 - \frac{1}{3}\Lambda e (\dot{U}^0)^{-2} U^2} + (\dot{V}^0 / \dot{U}^0) U + B \Theta(U) \sqrt{1 - \frac{1}{3}\Lambda e (\dot{U}^0)^{-2} U^2} \\ &\quad + (\dot{U}^0)^{-1} (\dot{Z}_i^0 A_i + \epsilon \dot{Z}_4^0 A_4) \Theta(U) U + C \Theta(U) U .\end{aligned} \quad (93)$$

The coefficients are

$$\begin{aligned}A_i &= \frac{1}{2} \left[H_{,i}(0) - \frac{1}{3}\Lambda Z_i^0 G(0) \right] , \quad A_4 = \frac{1}{2} \left[\epsilon H_{,4}(0) - \frac{1}{3}\Lambda Z_4^0 G(0) \right] , \quad B = \frac{1}{2}H(0) , \\ C &= \frac{1}{8} \left[H_{,2}^2(0) + H_{,3}^2(0) + \epsilon H_{,4}^2(0) + \frac{1}{3}\Lambda H^2(0) - \frac{1}{3}\Lambda \left(Z_p^0 H_{,p}(0) \right)^2 \right] ,\end{aligned} \quad (94)$$

in which $H(0) \equiv H(Z_p(0)) = H(Z_p^0)$, and the constants of integration are constrained by

$$\begin{aligned}-2\dot{U}^0\dot{V}^0 + (\dot{Z}_2^0)^2 + (\dot{Z}_3^0)^2 + \epsilon(\dot{Z}_4^0)^2 &= e , \\ -2U^0V^0 + (Z_2^0)^2 + (Z_3^0)^2 + \epsilon(Z_4^0)^2 &= \epsilon a^2 , \\ -U^0\dot{V}^0 - \dot{U}^0V^0 + Z_2^0\dot{Z}_2^0 + Z_3^0\dot{Z}_3^0 + \epsilon Z_4^0\dot{Z}_4^0 &= 0 .\end{aligned} \quad (95)$$

Interestingly, these general results also describe all geodesics in impulsive pp -wave spacetimes in a Minkowski universe. For $\Lambda = 0$ the geodesic equations (91) exactly reduce (omitting the coordinate Z_4) to the system which has been *rigorously* solved by Kunzinger and Steinbauer [47] using the Colombeau algebras. Of course, the solution (93), (94) reduces to the explicit form of geodesics presented in [47] and previous works [42], [90]. The same results are also obtained from the chaotic behaviour of geodesics in nonhomogeneous sandwich pp -waves [92]: in the impulsive limit the chaotic motion smears and becomes regular [93].

The main advantage of the approach presented above is that for *any* Λ this yields the system (91) which includes only “weakly” singular terms. In particular, there is no square of δ , contrary to other “direct” approaches, such as that presented in [53] which used the coordinates introduced in [42].

It follows from (93) that the geodesics are continuous but refracted by the impulse in the transverse directions Z_p . However, there is a discontinuity in the longitudinal coordinate V (and its derivative) on the impulse. The jump on $U = 0$ is given by $\Delta V = B = \frac{1}{2}H(0)$, which is in agreement with the Penrose junction conditions (18) in the “cut and paste” method. Further specific effects of nonexpanding impulses on privileged families of comoving observers associated with the natural coordinates (5) in the de Sitter and in the anti-de Sitter universe were discussed in [91] in detail. In particular, the behaviour of timelike and null geodesics (including their focusing) in the Hotta–Tanaka spacetime, a solution with a conformally flat pure radiation impulsive wave, and the Defrise-type impulse [94] were presented.

Note finally that symmetries of spacetimes with impulsive nonexpanding waves were studied for $\Lambda = 0$ in [95], and for $\Lambda \neq 0$ in [91]. There are always *at least three* Killing vector fields. In the metric form (36), these are null rotations generated by

$$\begin{aligned} x(\partial/\partial\mathcal{V}) + \mathcal{U}(\partial/\partial x) , \quad y(\partial/\partial\mathcal{V}) + \mathcal{U}(\partial/\partial y) , \\ [1 - \frac{1}{12}\Lambda(x^2 + y^2)](\partial/\partial\mathcal{V}) - \frac{1}{6}\Lambda\mathcal{U}[x(\partial/\partial x) + y(\partial/\partial y) + \mathcal{U}(\partial/\partial\mathcal{U})] , \end{aligned} \quad (96)$$

where $\eta = \frac{1}{\sqrt{2}}(x + iy)$. The Killing vectors (96) exactly reduce to those found previously by Aichelburg and Balasin [95] for impulsive pp -waves in the Minkowski background. Additional symmetries arise for specific forms of the structural function. For example, in case of the Hotta–Tanaka solution (59) the function H only depends on $z = \sqrt{1 - \epsilon a^{-2}(Z_2^2 + Z_3^2)}$. There is thus also an axial symmetry. Again, this is analogous to the axisymmetric Aichelburg–Sextl solution for the case $\Lambda = 0$ which also admits four Killing vectors [95].

4.3.2 Geodesics in expanding impulsive waves

As far as we know, an explicit form for geodesics in spacetimes with expanding spherical gravitational impulses has not been presented in the literature so far. It is possible to derive such geodesics assuming these are C^1 in the continuous coordinate system (30). With this assumption, the constants

$$\begin{aligned} Z_i \equiv Z(\tau_i) , \quad V_i \equiv V(\tau_i) , \quad U_i \equiv U(\tau_i) = 0 , \\ \dot{Z}_i \equiv \dot{Z}(\tau_i) , \quad \dot{V}_i \equiv \dot{V}(\tau_i) , \quad \dot{U}_i \equiv \dot{U}(\tau_i) , \end{aligned} \quad (97)$$

which characterize the positions and velocities at τ_i , the instant of interaction with the impulsive wave, must have the same values when evaluated in the limits $U \rightarrow 0$ both from the region in front ($U > 0$) and behind ($U < 0$) the impulse. Let us consider here only the case $\Lambda = 0$ for the sake of simplicity. We start with a general free motion

$$\begin{aligned} t^- &= \gamma \tau , \\ x^- &= \dot{x}_0 (\tau - \tau_i) + x_0 , \\ y^- &= \dot{y}_0 (\tau - \tau_i) + y_0 , \\ z^- &= \dot{z}_0 (\tau - \tau_i) + z_0 , \end{aligned} \quad (98)$$

where $\gamma = \sqrt{\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 - e}$, in the flat Minkowski region $U < 0$. It is straightforward to express the constants (97) in terms of the initial data (98) using the inverse to (19) and standard relations $\mathcal{U} = \frac{1}{\sqrt{2}}(t + z)$, $\mathcal{V} = \frac{1}{\sqrt{2}}(t - z)$, $\eta = \frac{1}{\sqrt{2}}(x + iy)$, as

$$\begin{aligned} Z_i &= \frac{x_0 + i y_0}{\gamma \tau_i - z_0} , \quad V_i = \frac{1}{\sqrt{2}} [(1 + \epsilon)\gamma \tau_i - (1 - \epsilon)z_0] , \\ \dot{Z}_i &= \frac{\dot{x}_0 + i \dot{y}_0}{\gamma \tau_i - z_0} - \frac{x_0 + i y_0}{(\gamma \tau_i - z_0)^2} \\ &\quad \times \frac{[(1 - \epsilon)\gamma - (1 + \epsilon)\dot{z}_0](\gamma \tau_i - z_0) + 2\epsilon(x_0 \dot{x}_0 + y_0 \dot{y}_0)}{(1 + \epsilon)\gamma \tau_i - (1 - \epsilon)z_0} , \\ \dot{V}_i &= \frac{1}{\sqrt{2}} \left\{ [(1 - \epsilon)\gamma - (1 + \epsilon)\dot{z}_0] [(1 - \epsilon)\gamma \tau_i - (1 + \epsilon)z_0] \right. \\ &\quad \left. + 4\epsilon(x_0 \dot{x}_0 + y_0 \dot{y}_0) \right\} [(1 + \epsilon)\gamma \tau_i - (1 - \epsilon)z_0]^{-1} , \\ \dot{U}_i &= \sqrt{2} \frac{x_0 \dot{x}_0 + y_0 \dot{y}_0 + z_0 \dot{z}_0 - \gamma^2 \tau_i}{(1 + \epsilon)\gamma \tau_i - (1 - \epsilon)z_0} . \end{aligned} \quad (99)$$

Now, we express the geodesics outside the impulse in the (locally) Minkowski space $U > 0$

$$\begin{aligned} \mathcal{V}^+ &= \dot{\mathcal{V}}_0^+ (\tau - \tau_i) + \mathcal{V}_0^+ , \\ \mathcal{U}^+ &= \dot{\mathcal{U}}_0^+ (\tau - \tau_i) + \mathcal{U}_0^+ , \\ \eta^+ &= \dot{\eta}_0^+ (\tau - \tau_i) + \eta_0^+ , \end{aligned} \quad (100)$$

using (26) as

$$\begin{aligned}
\mathcal{V}_0^+ &= AV_i, & \mathcal{U}_0^+ &= BV_i, & \eta_0^+ &= CV_i, \\
\dot{\mathcal{V}}_0^+ &= A\dot{V}_i - D\dot{U}_i + (A_{,Z}\dot{Z}_i + A_{,\bar{Z}}\dot{\bar{Z}}_i)V_i, \\
\dot{\mathcal{U}}_0^+ &= B\dot{V}_i - E\dot{U}_i + (B_{,Z}\dot{Z}_i + B_{,\bar{Z}}\dot{\bar{Z}}_i)V_i, \\
\dot{\eta}_0^+ &= C\dot{V}_i - F\dot{U}_i + (C_{,Z}\dot{Z}_i + C_{,\bar{Z}}\dot{\bar{Z}}_i)V_i,
\end{aligned} \tag{101}$$

in which the coefficients and their derivatives are given by the functions (27) evaluated at Z_i .

These general results can be used for a physical discussion of geodesics in specific impulsive solutions [96], such as the expanding spherical impulse generated by a snapping cosmic string which we have described above in sections 3.5.2 and 4.2.

5 Repetition and a few final remarks

In this essay, we have presented a brief review of exact solutions of Einstein's equations which describe impulsive waves in spaces of constant curvature. These are either gravitational and/or null matter non-expanding impulses, or expanding spherical impulsive purely gravitational waves (attached to cosmic strings) which propagate in Minkowski, de Sitter, or anti-de Sitter universes. We have systematically and explicitly described all the main methods for their construction: the “cut and paste” method, introduction of the continuous coordinate system, considering distributional limits of sandwich waves, geometrical embedding from higher dimensions, and boosts or limits of infinite acceleration of some initially static or accelerating sources. In the further part of the contribution we concentrated on some of their properties, in particular on the behaviour of geodesics. Physically interesting special solutions were also emphasized.

We have tried to provide an overall review. Nevertheless, we are aware of the fact that the point of view presented could depend on our own approach to the subject. The essay may thus be biased, as more space was certainly devoted to a description of those aspects of impulsive spacetimes which we had personally investigated. However, we attempted to present a comprehensive list of the relevant references. If some are missing this is unintentional, and the author deeply apologizes for this.

We have dedicated the essay to Professor Bičák on the occasion of the landmark anniversary of his birthday. It has been demonstrated that he contributed significantly also to the investigation of impulsive waves. Of course, this is just one small part of his both extensive and intensive work in the field of exact radiative spacetimes.

Let me finally close this contribution by personal words once more. It was Professor Bičák who put me wise to many of the secrets, mysteries, and wonders of Einstein's theory. He has also led my first steps in the world of relativity, formed and guided my professional activities. From the very beginning I have always admired his approach to general relativity. This is based on a deep understanding of the geometry and global structure of investigated spacetimes, yet with the emphasis being placed on presenting their clear physical interpretation. I hope that I have succeeded to emulate (at least partially) his “style” here. And I also hope that Jiří Bičák will not take his pencil during the reading of this essay, making many strokes, remarks, corrections, amendments, and improvements, as he has done so many times before with my previous manuscripts.

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